Edinburgh School of Economics

Discussion Paper Series

Number 84

Uniqueness of Equilibrium in Two-sided Matching

Simon Clark (University of Edinburgh)

Date
October 2002

Published by
School of Economics
University of Edinburgh
30 -31 Buccleuch Place
Edinburgh EH8 9JT
+44 (0)131 650 8361
http://www.ed.ac.uk/schools-departments/economics

THE UNIVERSITY *of* EDINBURGH

Uniqueness of Equilibrium in Two-sided Matching

Simon Clark

Edinburgh University

21 October 2002

Abstract

This paper analyses a sufficient condition for uniqueness of equilibrium in two-sided matching with non-transferable utility. The condition is easy to interpret, being based on the notion that a person's characteristics both form the basis of their attraction to the opposite sex, and determine their own sexual preferences.

Keywords: Uniqueness; matching; marriage.

JEL classification: C7.

I wish to thank Donald George, Ravi Kanbur, John Moore, and Jonathan Thomas for their helpful advice and comments.

1. Introduction

In 1962 Gale and Shapley posed and solved "the stable marriage problem", which asks whether it is possible to pair the members of one set (men) with members of another, disjoint, set (women), in such a way that no man and woman who are not paired with each other would both prefer to leave their partners and marry each other. Gale and Shapley proved that such an equilibrium, called a stable matching, exists and they showed how to find it.

This paper is concerned with the uniqueness of equilibrium. I propose and analyse a primitive condition on preferences that ensures a unique stable matching. The central idea is based on the notion that a person's characteristics, e.g. their physical appearance, their personal qualities, or their productive capabilities, both form the basis of their attraction to the opposite sex and determine who they are attracted to. The condition I propose is called the No Crossing Condition

(NCC). It has two components: firstly, we must be able to order completely any group of men, M, and any group of women, W, the implication being that such orderings are based on one or more personal characteristics which make up their type; secondly, men further along the ordering of M tend to prefer women further along the ordering of W,and *vice versa.* The exact sense of "tend to prefer" is made clear in the next section, but the NCC encompasses two special cases: when all members of one sex agree on their preferences for the other sex, and when each person would prefer a partner who is similar to themselves[1].

The next section of the paper sets up the formal matching framework, defines the NCC, and proves the central theorem of the paper. Section 3 considers the relationships between the No Crossing Condition and a condition recently proposed by Eeckhout (2000). Section 4 concludes.

2. Uniqueness of Stable Matching

2.1. The Matching Framework

The standard matching framework considers two finite and disjoint sets, both with n elements: a set of men M and a set of women W. We refer to $P = M \cup W$ as the population. Each man has complete, reflexive, and transitive preferences over the set W. We assume that these preferences are strict (so that no man is indifferent between two women), and are such that each man would rather be married to any woman than remain single. The preferences of a man $x \in M$ can thus be described by a binary relation $\succ_x$ defined on the set W, the statement $y \succ_x y'$ denoting that x prefers y to y'. Similar assumptions are made for women's preferences, *mutatis mutandis*, with the preferences of a woman $y \in W$ described by a binary relation $\succ_y$ defined on the set M. Let $\Phi = \{ \succ_i, i \in P\}$ be the preference profile (or set of preference relations) of the population $M \cup W$. The triple (M, W, Φ) constitutes a *marriage market.*

Definition 1 *A* matching μ *is a one-to-one function from* P *onto itself such that (i)* $x = \mu(y)$ *if and only if* $y = \mu(x)$; *(ii) if* $x \in M$ *then* $\mu(x) \in W$ *and if* $y \in W$ *then* $\mu(y) \in M$.

Definition 2 *A matching* μ *can be* blocked *by a pair* $(x, y) \in M \times W$ *for whom* $x \neq \mu(y)$ *if* $y \succ_x \mu(x)$ *and* $x \succ_y \mu(y)$. *A matching* μ *is* stable *if it cannot be blocked by any pair.*

Then we have:

Theorem 1 *A stable matching exists for every marriage market.*

Proof. See Gale and Shapley, 1962. ■

2.2. Ordering the sets M and W

The No Crossing Condition requires M and W to be ordered. The most straightforward way to approach this is to consider M and W when ordered as vectors (i.e. as ordered lists of the elements of M and W) with different orderings represented by different vectors. Let $I_n = \{1, 2, ..., n\}$.

Definition 3 *The vector* $m = (m_i)$ *is an ordering of* M *if (i)* m *has* n *elements; (ii) for all* $i \in I_n$, $m_i \in M$; *(iii) for all* $x \in M, x = m_i$ *for some* $i \in I_n$; *similarly the vector* $w = (w_i)$ *is an ordering of* W *if (i)* w *has* n *elements; (ii) for all* $i \in I_n$, $w_i \in W$; *(iii) for all* $y \in W, y = w_i$ *for some* $i \in I_n$.

We can now move easily from one ordering of M or W to another; this will be useful when we analyse the relationship between the No Crossing Condition and the condition proposed by Eeckhout (2000).

2.3. The No Crossing Condition

This may now be stated quite simply:

Definition 4 *A population $P = M \cup W$ with preference profile Φ satisfies the No Crossing Condition (NCC) if there exists an ordering m of M and an ordering w of W such that if $i < j$ and $k < l$ then*
(i) it is not the case that both $m_l \succ_{w_i} m_k$ and $m_k \succ_{w_j} m_l$;
(ii) it is not the case that both $w_j \succ_{m_k} w_i$ and $w_i \succ_{m_l} w_j$.

It is sometimes convenient to refer to the orderings m and w themselves as satisfying the NCC. Part (i) of the definition may be interpreted as saying that it cannot be the case that the woman further back in the female ordering (w_i) prefers the man further forward in the male ordering (m_l) and at the same time woman further forward in the female ordering (w_j) prefers the man further back in the male ordering (m_k). Diagrammatically this rules out the preferences depicted in Figure 1, where the sexes are ordered along the two horizontal lines, and the arrow from each woman points to the man she prefers out of the two shown. If the NCC is satisfied there exist orderings m and w such that for any pair of women and any pair of men the two arrows representing the women's preferences do not cross, nor do the two arrows representing the men's preferences.

The condition does not rule out the possibility that both women prefer the same man; more generally it allows for all members of one sex to have the same

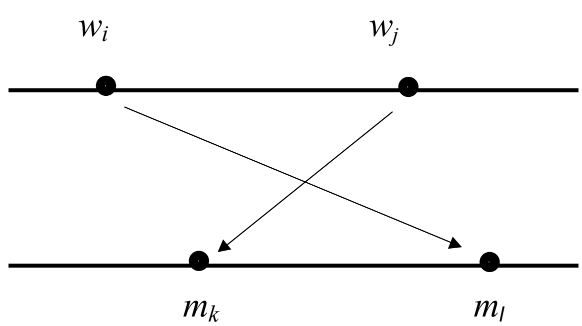

Figure 1: Preferences ruled out by the No Crossing Condition

preferences. Nor does it necessarily imply 'single-peakedness' of preferences", given the orderings m and w. For example if $n = 3$ the NCC does not forbid preferences for w_1 such that $m_3 \succ_{w_1} m_1 \succ_{w_1} m_2$. However in this case the NCC rules out either $m_1 \succ_{w_2} m_3$ or $m_2 \succ_{w_2} m$, although it has nothing to say about w_2's ranking of m_1 and m_2.

A very important property of the NCC is that if it holds for a population P then it also holds for any sub-population of P with equal numbers of men and women. Formally:

Lemma 1 *Let the population $P = M \cup W$ satisfy the No Crossing Condition and let $M' \subset M$ and $W' \subset W$, where $\#(M') = \#(W') = n'$; then the population $P' = M' \cup W'$ satisfies the No Crossing Condition.*

Proof. *See Appendix* ■

2.3.1. No Crossing and the Existence of Fixed Pairs

We now develop two lemmas that lie at the heart of the main theorem on uniqueness. If we can find any couple who love each other (where "love" means "prefer, out of the members of the opposite sex) then each such couple, called a *fixed pair,* must be matched in equilibrium. The main theorem then uses the No Crossing Condition to identify a sequence of n fixed pairs, thus generating a unique equilibrium matching.

Definition 5 *A couple* $(x, y) \in M \times W$ *is a fixed pair of the population* $P = M \cup W$ *if* $y \succ_x y'$ *for all* $y' \in W \backslash y$ *and* $x \succ_y x'$ *for all* $x' \in M \backslash x$*.*

The advantage of being able to identify fixed pairs is that any stable matching of P must consist of the partnerships of the fixed pairs of P plus a stable matching of the remainder of the population. Formally.

Lemma 2 *Let* μ *be a stable matching of the population* $P = M \cup W$ *which has* p *fixed pairs* $(x_i, y_i), i \in I_p$*; let* $P' = M' \cup W'$*, where* $M' = M \backslash \{x_1, x_2, ..., x_p\}$ *and* $W' = W \backslash \{y_1, y_2, ..., y_p\}$; *and let* μ' *be a matching of the population* P' *defined by* $\mu'(z) = \mu(z)$ *for all* $z \in P'$ *. Then (i)* $\mu(x_i) = y_i$ *for all* $i \in I_p$*; (ii)* μ' *is a stable matching of* P'*.*

Proof. *See Appendix* ■

The problem of finding a unique stable matching of the population P can thus be broken down into finding the fixed pairs of P, and then finding a unique stable matching of P'. But this requires that P does indeed have at least one fixed pair: enter the No Crossing Condition.

Lemma 3 *If a population* $P = M \cup W$ *satisfies the No Crossing Condition, then it has a fixed pair.*

Lemma 4* *Proof. *See Appendix.* ■

The proof considers the function that for each man x gives x's *rival*, the preferred man of x's preferred woman. Given orderings m and w satisfying the NCC, this function is non-decreasing in the sense that if x is further along the ordering m then x's rival can be no further back in the ordering. The existence of a fixed point, and hence of a fixed pair, is almost immediate. But there is no reason to suppose that if m_i and w_k are a fixed pair then $i = k$. For example, the shortest man and the tallest woman form a fixed pair if everyone prefers a partner as tall as themselves and all men are taller than all women. Of course, the fixed point is not necessarily unique.

2.4. The Main Theorem

Bringing together Lemmas 4, 5, and 6, we now have the main uniqueness result:

Theorem 2 *If a population $P = M \cup W$ satisfies the No Crossing Condition then there exists a unique stable matching*

Proof. *See Appendix* ■

The proof shows how to construct the unique stable matching: first match the fixed pairs of P; take the remaining population P_2, match the fixed pairs of P_2; and so on, until the population is exhausted (perhaps literally). To illustrate how the successive identification of fixed pairs leads to a unique stable matching, consider the following example of a population of women with heights 1.50, 1.64, 1.69, 1.78, and men with heights 1.60, 1.67, 1.72, 1.80. Each individual would prefer to be matched with someone as near to their own height as possible i.e. someone of height h_1 matched with someone of height h_2 has utility that is a negative function of $|h_1 - h_2|$ .Such preferences satisfy the No Crossing Condition,

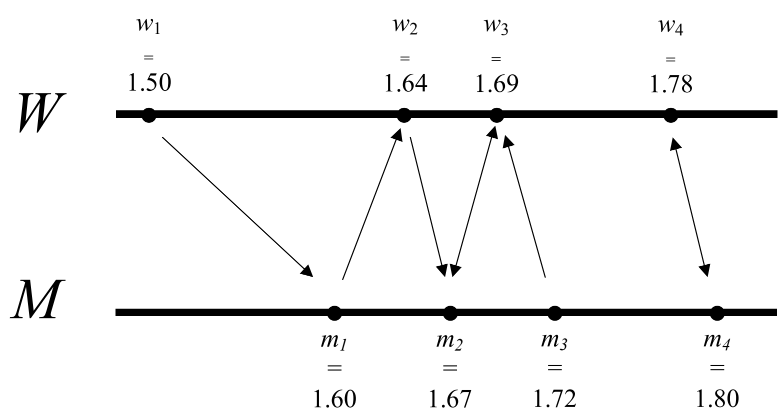

Figure 2: Preferences to illustrate Theorem 2

and are illustrated in Figure 2 (the arrows from each person point to her/his most preferred partner.

The fixed pairs of this population are (w_3, m_2) and (w_4, m_4), so they are matched in any stable matching. The remaining population, $P_2 = W_2 \cup M_2$, equals $\{w_1, w_2, m_1, m_3\}$, with heights 1.50, 1.64, 1.60, and 1.72 respectively. P_2 also satisfies the No Crossing Condition and the couple (w_2, m_1) are a fixed pair; note that w_2 would have preferred m_2 but he is already matched with w_3. Finally, w_1 and m_3 are left and must be matched. This process by which new fixed pairs emerge as others are taken out of the population is illustrated in Fig 3, where the bold double arrows denote a fixed pair of the population or sub-population under consideration.

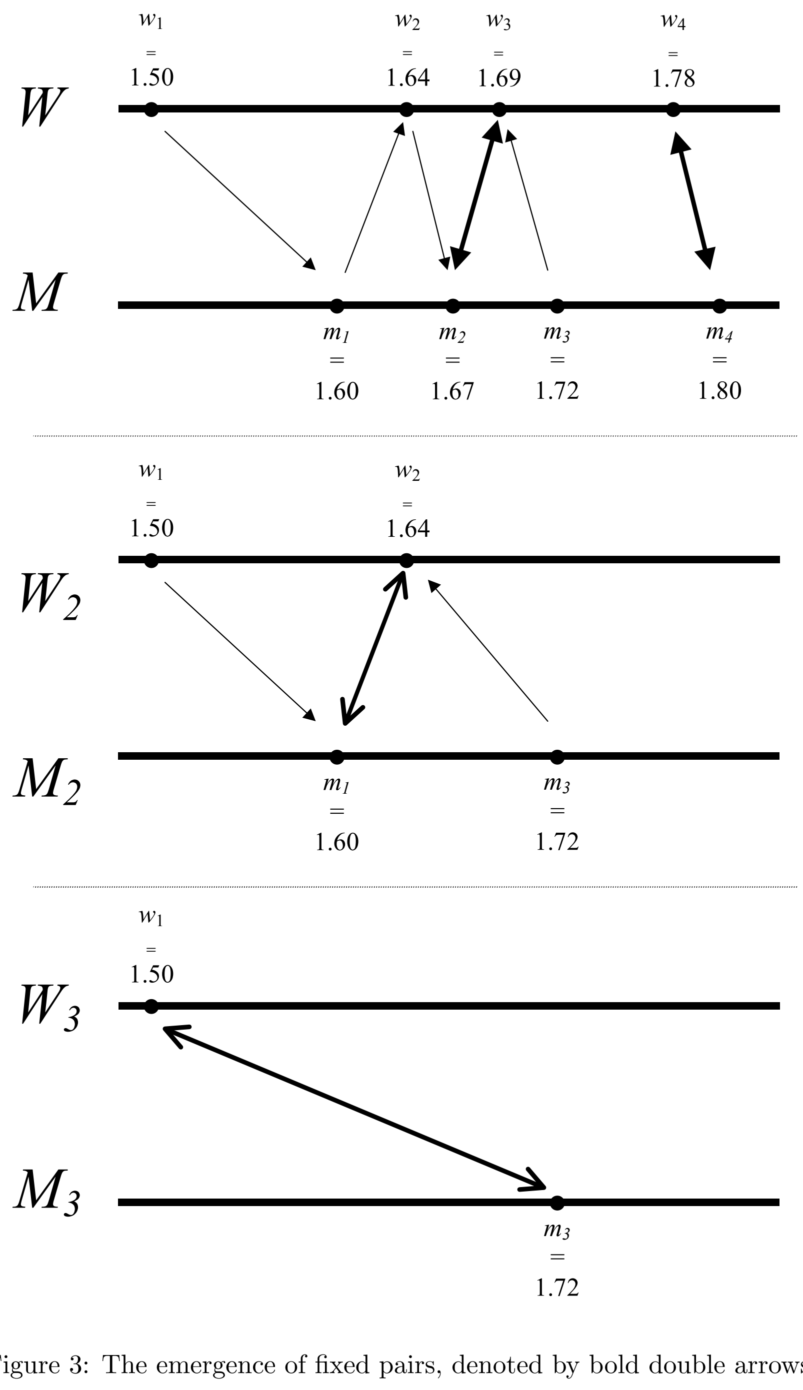

Figure 3: The emergence of fixed pairs, denoted by bold double arrows.

3. No Crossing and Sequential Preferences

Recently, Eeckhout (2000) has suggested a sufficient condition for uniqueness. Let m and w be orderings of M and W respectively, not necessarily satisfying the NCC. Suppose that, for $i < n$, m_i prefers w_i to all the women from w_{i+1} to w_n, and w_i prefers m_i to all the men from m_{i+1} to m_n. I call this the Sequential Preference Condition (SPC), and for convenience refer to the orderings m and w as satisfying the SPC. Then there is a unique stable matching in which m_i is matched with w_i, for all i (Theorem 1 in Eeckhout, 2000). The equilibrium can be constructed by a sequential process: m_1 and w_1 prefer each other above all others and so must be paired in any stable matching (since they could block any matching in which they were not paired); m_2 and w_2 prefer each other to anyone else in $W \backslash w_1$ and $M \backslash m_1$ respectively and hence must also be paired in equilibrium (since they could block any matching in which they were not paired but m_1 and w_1 were paired); and so on, until we are left with m_n and w_n, who would rather marry each other than remain single.

A drawback of the Sequential Preference Condition is that it does not indicate when or why it might be satisfied in any particular population. It applies a test to the preference orderings of a given population, but gives no clue about the underlying structure of tastes that might result in the test being satisfied. Consequently whether the SPC holds or not depends critically on the exact membership of the sets W and M. For example, if we reduce both the number of men and the number of women by one, the condition may no longer hold (unless we take out the i^{th} man and the i^{th} woman). This possibility is illustrated in Table 1, which gives the preferences of a population of three men and three women:

agent	1st preference	2nd preference	3rd preference
m_1	w_1	w_2	w_3
m_2	w_2	w_3	w_1
m_3	w_1	w_2	w_3
w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_2	m_1

Table 1

The SPC is satisfied and m_i is matched with w_i, $1 = 1, 2, 3$. But if we take out m_1 and w_2 and consider the population of $M' = \{m_2, m_3\}$ and $W' = \{w_1, w_3\}$ then w_1 prefers m_2 who prefers w_3 who prefers m_3 who prefers w_1, with the result that both of the possible matchings are stable.

What is the relationship between the NCC and Eeckhout's SPC? Let m and w be orderings satisfying the NCC for a population $P = M \cup W$. Recall that P satisfies the SPC if there exists orderings m' and w' such that for $i < n$, m'_i prefers w'_i to all the women from w'_{i+1} to w'_n, and w'_i prefers m'_i to all the men from m'_{i+1} to m'_n. If P satisfies the NCC, orderings m' and w' satisfying the SPC can be derived from the order in which the fixed pairs of P and its subpopulations are generated in constructing the unique stable matching of P. In essence, the k^{th} elements of m' and w' must be the k^{th} fixed pair in that sequence. More precisely, consider the sequence of populations $\{P_s\}, s = 1, ..., S$ such that $P_{s+1} = P_s \backslash F_s$, with $P_1 = P$, where F_s is the set of individuals who constitute the fixed pairs of P_s. Each man in M and each woman in W is an element of only one set in the sequence $\{F_s\}$. Let $n_s = \#F_s/2$ and $N_s = \sum_{t=1}^{s} n_t$ (so that $N_S = n$) and set $N_0 = 0$. Then the vectors m' and w' satisfy the SPC if (i) they are orderings of

M and W respectively and (ii) for any $k \leq n$, $m'_k = m_i$ and $w'_k = w_j$ for some i, j such that m_i and w_j are a fixed pair of P_s, where s is uniquely defined by the condition $N_{s-1} < k \leq N_s$.

In short, the NCC implies the SPC, for a given population P. The reverse is not true, as the example of Table 1 shows; the population $P = M \cup W$ satisfies the SPC, but not the NCC; if P did satisfy the NCC then so would its subset $P' = M' \cup W'$, which would ensure a unique stable matching of P'.

An alternative view of the relationship between the NCC and the SPC is to regard M and W as the sets of all possible men and women. In any particular instance, we are therefore dealing with subsets M' and W'. From Lemma 1, if $P = M \cup W$ satisfies the NCC, so does $P' = M' \cup W'$, so it is sufficient, when analysing the population P', to show or assume that P satisfies the NCC. Moreover, when considering the orderings m' and w' of M' and W' that satisfy the NCC, the order in which any two men or two women appear in m' or w' is independent of the other elements in those vectors. In effect, the orderings m and w of M and W that satisfy the NCC may be treated as "master orderings". But consider now the emergence of fixed pairs as the stable matching is constructed (for example, as in Fig.3). The positions in m' and w' of the first fixed pair, the second fixed pair, and so on, may bear no resemblance to the order in which they emerge, and will typically vary with the precise membership of M' and W'. But it is the order in which they emerge as fixed pairs that gives the orderings that satisfy the SPC. Both theoretically and from an applied perspective, it seems an advantage that the ordering satisfying the NCC is invariant to the particular groups M' and W' being considered.

4. Conclusion

Uniqueness of equilibrium is a typically regarded as a desirable characteristic of an economic model. It helps to make prediction and comparative statics sharp and unambiguous. This paper shows that the standard model of two-sided matching has a unique stable matching if agents' preferences satisfy a condition, that is both intuitively reasonable and easy to interpret, being based on the notion that a person's characteristics both form the basis of their attraction to the opposite sex, and determine their own sexual preferences. If we are prepared to assume that men and women can be ordered on the basis of their characteristics, and that men further along the male ordering tend to prefer women further along the female ordering, then the No Crossing Condition is satisfied and equilibrium is unique.

One application of the results of this paper may be found in Clark and Kanbur (2002), which looks at the question of assortment in two-sided matching. One possible interpretation of the NCC is that agents tend to prefer partners who are similar to themselves. It might therefore be thought that in the equilibrium matching like will match with like, resulting in positive assortment. Clark and Kanbur show that this line of reasoning is incorrect and, using the NCC, show how the degree of assortment depends on, *inter alia,* the distribution of agents characteristics. They thus demonstrate how information about utility functions can be used to see if the NCC is satisfied, regardless of the precise membership of the two groups to be matched.

Appendix

Proof of Lemma 1. Take the orderings m and w that satisfy the NCC for the population P, delete those elements corresponding to $M\backslash M'$ and $W\backslash W'$ to form the n' dimensional vectors $m'$and w'. Then since m' and w' must continue to satisfy conditions (i) and (ii) in Definition 4 they are orderings that satisfy the NCC for the population P'. ■

Proof of Lemma 2. If (i) is not satisfied then μ can be blocked by one of the fixed pairs (x_i, y_i) $i \in I_{n'}$ and hence cannot be stable, a contradiction. If (i) is satisfied but not (ii), then there exists a pair $(x', y') \in M' \times W'$, with $x' \neq \mu'(y)$, who can block the matching μ' i.e. $y' \succ_{x'} \mu'(x')$ and $x' \succ_{y'} \mu'(y')$. The definition of μ' ($\mu'(z) = \mu(z)$ for all $z \in P'$) implies that $\mu'(x') = \mu(x')$ and $\mu'(y') = \mu(y')$ for all $y' \in W'$, so that $y' \succ_{x'} \mu(x')$ and $x' \succ_{y'} \mu(y')$. This means that the pair (x', y') can block the matching μ,and hence μ cannot be stable, a contradiction.

Proof of Lemma 3. For any man $x \in M$, let $f(x) \in W$ denote his preferred woman in W; i.e. $f(x) \succ_x y$ for all $y \in W\backslash f(x)$. Since preferences are complete and strict, $f(x)$ exists and is unique. Similarly, for any woman $y \in W$, let $g(y) \in M$ denote her preferred man in M; i.e. $g(y) \succ_y x$ for all $x \in W\backslash g(y)$. $g(y)$ also exists and is unique. Let m and w be orderings of M and W satisfying the NCC for $P = M \cup W$. For each element of m the function f specifies an element of w; this in turn defines a function $\phi : I_n \to I_n$ as follows: if $f(m_k) = w_i$ then $\phi(k) = i$, which may be read as "the k^{th} man prefers the i^{th} woman". Compare the preferences of m_k and m_l, where $k < l$. If $f(m_k) = f(m_l)$ then $\phi(k) = \phi(l)$. If $f(m_k) \neq f(m_l)$, then $\phi(k) \neq \phi(l)$ and $w_{\phi(k)} \succ_{m_k} w_{\phi(l)}$; but if $\phi(l) < \phi(k)$ then part (ii) of Definition 4 of the NCC, with $i = \phi(l)$ and $j = \phi(k)$, implies that $w_{\phi(k)} \succ_{m_l} w_{\phi(l)}$, a clear contradiction. Hence if $k < l$ then $\phi(k) \leq \phi(l)$ i.e. the function ϕ is non-decreasing. A similar argument applies to the function $\gamma : I_n \to I_n$, defined as follows: if $g(w_i) = m_k$ then $\gamma(i) = k$; if $i < j$ then $\gamma(i) \leq \gamma(j)$ i.e. the function γ is non-decreasing. Now, consider the composition of ϕ and γ, the function $\rho(k) = \gamma(\phi(k))$; this gives the position, in the ordering m, of the preferred man of the k^{th} man's preferred woman (the k^{th} man's *rival)*; i.e. $m_{\rho(k)} = g(f(m_k))$. Since ϕ and γ both map I_n into I_n and are non-decreasing the function ρ also maps I_n into I_n and is non-decreasing. It therefore has a fixed point $k^* = \rho(k^*)$. Let $i^* = \phi(k^*)$; then $k^* = \gamma(i^*)$; thus $w_{i^*} = f(m_{k^*})$ and $m_{k^*} = g(w_{i^*})$; i.e. m_{k^*} and w_{i^*} are a fixed pair. ■ ■

Remark. The proof of Lemma 3 could have proceeded by considering the composition of γ and ϕ, the function $\kappa(i) = \phi(\gamma(i))$ (the i^{th} woman's rival*)* . Clearly if k^* is a fixed point of ρ, then $i^* = \phi(k^*)$ is a fixed point of κ.

Proof of main theorem. We consider a sequence of populations $\{P_s\}, s = 1, ..., S$ such that $P_{s+1} = P_s \backslash F_s$, with $P_1 = P$, where F_s is the set of individuals who constitute the fixed pairs of P_s; i.e. if (x, y) is a fixed pair of P_s then $\{x, y\} \subseteq F_s$. S is defined by the condition $P_S = F_S$. Since F_s is unique given P_s, the sequence $\{P_s\}$ is uniquely defined. By the repeated application of Lemmas 1 and 3, each element in the sequence $\{P_s\}$ satisfies the No Crossing Condition and has at least one fixed pair. Thus P_{s+1} is a proper subset of P_s, and since P is finite S exists and is finite.

Let μ be any stable matching of P, and for all $s \in I_S$, let μ_s be a matching of P_s defined by $\mu_s(z) = \mu(z)$ for all $z \in P_s$; i.e. μ_s is the matching μ as it applies to the population P_s. Lemma 2 , part (ii), says that if μ_s is a stable matching of P_s then μ_{s+1} is a stable matching of P_{s+1}. But since $\mu = \mu_1$is a stable matching of $P = P_1$, this implies that for all $s \in I_S$, μ_s is stable,. Then by Lemma 2, part (i), for all $s \in I_S$, $x = \mu_s(y)$ for any fixed pair (x, y) of P_s; and hence, given the definition of μ_s, we have $x = \mu(y)$ for any fixed pair (x, y) of P_s and for all $s \in I_S$. But since $P = \cup_{s=1}^{S} F_s$ every individual in P is a member of some fixed pair of some population P_s and is therefore matched by μ with the other member of the fixed pair. Since the sequence $\{P_s\}$ and the associated sequence$\{F_s\}$ are independent of the choice of which stable matching μ of P we consider, the matching μ is uniquely determined. ■

References

Clark, S. and R. Kanbur, (2002), "Stable partnerships, matching, and local public goods", Working Paper, Department of Economics, Edinburgh University.

Eeckhout, J. (2000), "On the uniqueness of stable marriage matchings", Economics Letters, 69,1-8.

Gale, D. and L. Shapley, (1962), "College admissions and the stability of marriage", American Mathematical Monthly, 69, 9-15.

Notes

[1]Clark and Kanbur (2002) analyse these two special cases based on preferences for local public goods. A central issue is whether the equilibrium matching displays positive assortment. It does when all members of one sex agree on their preferences

for the other sex, but when each person would prefer a partner who is similar to themselves positive assortment is only one of many possibilities