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Abstract

In this paper we propose a simple testing procedure to detect the presence of nontstationarity against nonlinear stationarity based on the Smooth Transition Autoregressive modelling approach. We provide an advance over the existing literature in three senses. *First,* we derive the limiting nonstandard distribution of the proposed $NLADF$ tests. *Second*, we establish the superior power performance of the $NLADF$ test over the standard linear ADF test under the alternative of nonlinear stationarity via Monte Carlo simulation exercises. *Third*, we provide an application to *ex post* real interest rates from six major OECD countries, and find the $NLADF$ test is able to reject a unit root in cases, whereas the linear ADF tests fail.

JEL Classification: C12, C13, C32.
Key Words: Exponential Smooth Transition Autoregressive Model, Nonlinearity, Unit Root t Statistics, Critical Values, Monte Carlo Simulations, Real Interest Rates.

1 Introduction

There is a growing dissatisfaction with the standard linear ARMA framework which investigators use to test for unit roots. Much of this arises from the fact that in several areas of economics a theoretical prediction of stationarity is confounded in practice by the persistent failure of the standard ADF test to reject the null of a unit root. For example, in international monetary economics the regular finding of a unit root in real exchange rates causes discomfort to economists who wish to build models around a long run purchasing power parity relationship (*e.g.* Taylor and Peel, 1998). Another notable area of discomfort occurs in macro-finance where apparent unit root behaviour in real interest rates violates mean reversion in the aggregate marginal product of capital and makes the adherence to simple constant returns to scale production functions used in modern macroeconomic growth and RBC theory hard to maintain (*e.g.* Rose, 1988).

Some authors have accepted the results of the tests and have reformulated the economic theory. For example, Edison and Kloveland (1987) point out that whilst the homogeneity postulate behind the standard view of purchasing power parity (PPP) is only likely to hold in the long run, long runs of data may encounter regime changes in tastes and technology which in turn imply permanent movements in the terms of trade. They find that adjusting for "general equilibrium" shocks, enables them to reject the unit root in real exchange rates and provide support for the PPP hypothesis.

Increasingly though, investigators are looking to alternative frameworks within which to test for unit roots. The literature here has two branches. The first focuses on the use of panel data and its role in increasing the power of standard unit root tests. A good example is Abuaf and Jorion (1990), who use a panel data test to reject the joint hypothesis of unit roots in each of a group real exchange rates against an alternative that they are all stationary. For a general econometric discussion see Im *et al.* (1997). The second branch uses alternative forms of stationarity to simple AR or ARMA models such as fractional integration (*e.g.* Mills, 1993) and nonlinear transition dynamics (*e.g.* Pesaran and Potter, 1997).

In this paper we analyse the implications of the existence of a particular kind of nonlinear dynamics for unit root testing procedures. We aim to bridge the two areas of nonlinearity and nonstationarity by investigating ways to distinguish nonstationary linear systems from stationary nonlinear ones. Recently, Caner and Hansen (1998) have attempted to address similar issues in the context of a threshold autoregressive model. See also Pippenger and Goering (1993), Balke and Fomby (1997), Michael, Nobay and Peel (1997), Enders and Granger (1998), O'Connell (1998) and Kapetanious and Shin (2000) for some examples of the growing literature examining the interplay between nonstationarity, cointegration and nonlineariy.

Here we provide an alternative framework by analysing a test of the null of a unit root process against a nonlinear Exponential Smooth Transition AutoRegressive ($ESTAR$) model alternative (Granger and Teräsvirta, 1993 and Teräsvirta, 1994). To this end, we develop the conventional t-test procedure and call the resulting statistic the nonlinear ADF ($NLADF$) test. The current paper provides an advance over the existing literature in three senses. *First,* we derive the limiting distribution of the suggested $NLADF$ tests. These distributions are non-standard, but different from those of the linear ADF tests as expected. *Second*, we conduct some Monte Carlo simulation exercise and examine small sample size and power performance of the $NLADF$ tests. We find that the linear ADF tests suffer from some upward size distortion in small samples whereas the size of the $NLADF$ test rarely rises above the nominal 5% level. In addition, under the alternative of an $ESTAR$, the $NLADF$ test has superior power in most of cases considered. *Third*, we provide an application to *ex post* real interest rates from six major OECD countries, which demonstrates the empirical worth of our approach. In particular, our $NLADF$ test is able to reject a unit root in cases, whereas the linear ADF tests fail. In addition, the goodness of fit is substantially better under the $ESTAR$ alternative than it is under the simple AR alternative in nearly all of the country rates considered.

The plan of the paper is as follows: Section 2 sets out the model, develops the $NLADF$ test statistics, derives their asymptotic distributions and provides relevant asymptotic critical values. The modification of the test is also provided in presence of serial correlation. Section 3 addresses the issue of the small sample performance of the proposed $NLADF$ tests using a limited set of Monte Carlo experiments. It compares the size and power performance of the standard linear ADF testing procedure with the new tests that take account of the specific nonlinear nature of the alternative. Section 4 presents an empirical illustration of our proposed test. Section 5 contains some concluding remarks. Mathematical proofs are collected in an appendix.

2 Testing for a Unit Root Against the Nonlinear $STAR$ Model

Consider a univariate smooth transition autoregressive of order 1 ($STAR(1)$) model,

$$y_t = \beta y_{t-1} + \beta^* y_{t-1} \Phi\left(\theta; y_{t-d}\right) + \varepsilon_t,\ t = 1, ..., T;\ d \geq 1, \tag{2.1}$$

where $\varepsilon_t \sim iid\left(0, \sigma^2\right)$, and β, β^* are unknown parameters. We assume that y_t is a mean zero stochastic process. (For the case with non-zero mean and/or with a linear time trend see the discussion after Theorem below.) $\Phi\left(\theta; y_{t-d}\right)$ denotes the transition function. We also assume that $\theta \geq 0$, and the delay parameter $d \geq 1$ is given. If θ is positive, then it effectively determines the speed of mean reversion. The representation (2.1) makes economic sense in that many economic models predict that the underlying system tends to display a dampened behavior towards an attractor when it is (sufficiently far) away from it, but that it shows some instability within the locality of that attractor. A classic example is the floor and ceiling model of output analysed by Hicks (1950). In assets markets too there are applications. When the differential between the risk adjusted returns on two assets is wide, the profitability of "arbitrage" is higher than when this differential is low due to the existence of fixed transactions costs. As a result the speed of reversion to equilibrium, *i.e.* the speed with which returns are equalised varies inversely with the size of the differential itself. For recent theoretical works based on international transaction costs see, for example, Sercu *et al.* (1995).

Following the literature on the $STAR$ models here we consider one popular transition function: the exponential function,

$$\Phi_E(\theta y_{t-d}) = 1 - \exp\left(-\theta y_{t-d}^2\right). \tag{2.2}$$

The exponential transition function is bounded between zero and 1, *i.e.* $\Phi : \mathbb{R} \to [0,1]$, has the properties

$$\Phi_E(0) = 0; \lim_{x\to\pm\infty} \Phi_E(x) = 1,$$

and is symmetrically U-shaped around zero.[1]

Using (2.2) in (2.1) we obtain an exponential $STAR$ ($ESTAR$) model

$$y_t = \beta y_{t-1} + \beta^* y_{t-1}\left[1 - \exp\left(-\theta y_{t-d}^2\right)\right] + \varepsilon_t. \tag{2.3}$$

We consider a null hypothesis of a unit root which in terms of the above model implies that $\beta = 1$ and $\theta = 0$ (and thus $\Phi_E(\cdot) = 0$). Under the null, then (2.3) becomes the nonstationary linear $AR(1)$ model:

$$y_t = \beta y_{t-1} + \varepsilon_t. \tag{2.4}$$

Under the alternative of stationarity, θ is strictly positive and (2.3) becomes

$$y_t = \{\beta + \beta^*\Phi_E(\theta y_{t-d})\} y_{t-1} + \varepsilon_t,\ 0 < \Phi_E(\theta y_{t-d}) < 1. \tag{2.5}$$

This clearly shows that y_t would locally follow a unit root in the region of $y_{t-d} = 0$ in which case $\beta + \beta^*\Phi_E(\theta y_{t-d})$ would be unity. Large values of y_{t-d} on the other hand would result in an approximately linear $AR(1)$ process with the stable root $\beta + \beta^*$ provided that $-2 < \beta^* < 0$. We assume that the latter is the case.

Explicitly then we wish to test

$$H_0 : \theta = 0, \tag{2.6}$$

against the alternative[2]

$$H_1 : \theta > 0. \tag{2.7}$$

Obviously, testing the null hypothesis (2.6) directly is not feasible, since β^* is not identified under the null (see Davies (1977, 1987)).

Considering that the standard linear ADF test is not expected to be very powerful when the true process is stationary but nonlinear, we now develop the direct testing framework. Imposing $\beta = 1$ in (2.5) and assuming $d = 1$ without loss of generality, we rewrite the $ESTAR$ model (2.3) as[3]

$$\Delta y_t = \beta^* y_{t-1}\left\{1 - \exp\left(-\theta y_{t-1}^2\right)\right\} + \varepsilon_t. \tag{2.8}$$

[1]Alternatively, we may consider the use of the logistic function:

$$\Phi_L(\theta y_{t-d}) = \frac{2}{1 + \exp(-\theta y_{t-d})} - 1,$$

which is bounded between 0 and ± 1.

[2]Under the alternative (2.7) y_t is stable as $1 + \beta_*\Phi_E(\theta y_{t-d}) < 1$ for all y_{t-d}.

[3]Our modelling approach is clearly different from Caner and Hansen's (1998) study based on a stationary threshold autoregressive model at least in the following two senses: *First*, we choose the transition variable as y_{t-1}^2, while they use Δy_{t-1} as a threshold variable. The choice of y_{t-1}^2 is more sensible in terms of the speed of convergence arguments given in the text above. *Second*, we subject only the coefficient on the lagged dependent variable to the threshold nolinearity which is parameter of main interest (see also (2.12) below), while they adopt the same nonlinear scheme for all of their parameters.

We now define the following t-test statistic for (2.6) by

$$NLADF = \frac{\widehat{\delta}}{s.e.\left(\widehat{\delta}\right)}, \tag{2.9}$$

where $\widehat{\delta}$ is the OLS estimate obtained from the following auxiliary regression:[4]

$$\Delta y_t = \delta y_{t-1}^3 + error, \tag{2.10}$$

and $s.e.\left(\widehat{\delta}\right)$ is its standard error of $\widehat{\delta}$. Our test is motivated by the fact that the auxiliary regression is testing the significance of the score vector from the quasi-likelihood function of the $ESTAR$ model, evaluated at $\theta = 0$. The LM test of (2.6) against (2.7) also tests the significance of this term and is thus intimately related to the t-test that we consider.[5]

Unlike the case of testing the linearity against the nonlinearity for the stationary process the $NLADF$ test does not have an asymptotic standard normal distribution.

Theorem 2.1 *Under the null (2.6) the NLADF statistic defined by (2.9) has the following asymptotic distribution:*

$$NLADF \Rightarrow \frac{\left\{\frac{1}{4}W(1)^4 - \frac{3}{2}\int_0^1 W(r)^2 dr\right\}}{\sqrt{\int W(r)^6 dr}}, \tag{2.11}$$

where $W(r)$ is the standard Brownian motion defined on $r \in [0,1]$.

Proof. See the Appendix. ■

The $NLADF$ test for the above simple case needs only minor modification to accommodate the process with a non-zero mean and a linear trend. *First*, in the case where the data has non-zero mean such that $x_t = \mu + y_t$, we use the de-meaned data $x_t - \bar{x}$ in (2.1), where $\bar{x}$ is the sample mean. In this case the asymptotic distributions are the same as (2.11) except that $W(r)$ is replaced by the de-meaned standard Brownian motion $\widetilde{W}(r)$ defined on $r \in [0,1]$. Similarly, for the case with non-zero mean and non-zero linear trend such that $x_t = \mu + \delta t + y_t$ we use the de-meaned and de-trended data $x_t - \hat{\mu} - \hat{\delta} t$ in (2.1), where $\hat{\mu}$ and $\hat{\delta}$ are the OLS estimators of μ and δ. Now the associated asymptotic distributions are such that $W(r)$ is replaced by the de-meaned and de-trended standard Brownian motion $\widehat{W}(r)$ defined on $r \in [0,1]$.

Asymptotic critical values of the $NLADF$ statistics for the above three cases, denoted $NLADF1$, $NLADF2$, $NLADF3$, respectively, have been tabulated via stochastic simulations with $T = 1,000$ and 100,000 replications, and presented in Table 1 below.

[4] The Talyor series approximation is used in deriving (2.10) from (2.8), see Granger and Teräsvirta (1993) and Teräsvirta (1994) for more details.

[5] To be specific, LM would be (asymptotically) equal to the square of the t-ratio defined by (2.9). The underlying error variance in (2.10) is estimated under the null for the LM test, while it is estimated under the alternative for the t test. We are mainly interested in one sided alternative of stationarity, and thus expect the one-sided t-test procedure considered here to be more powerful.

Table 1. Asymptotic Critical Values			
fractile	$NLADF1$	$NLADF2$	$NLADF3$
1%	-2.82	-3.48	-3.93
5%	-2.22	-2.93	-3.40
10%	-1.92	-2.66	-3.13

Theorem 2.2 *Under the alternative hypothesis (2.7) with the ESTAR model (2.8), the $NLADF$ statistic defined by (2.9) is consistent.*

Proof. See the Appendix. ■

2.1 Modification in presence of serial correlation

We now consider the more general case where the errors in (2.8) are serially correlated. Following Dickey and Fuller (1979) we extend the simple model and consider the following nonlinear $ADF\,(p)$ regression:[6]

$$\Delta y_t = \sum_{j=1}^{p} \phi_j \Delta y_{t-j} + \beta^* y_{t-1} \left\{1 - \exp\left(-\theta y_{t-1}^2\right)\right\} + \varepsilon_t, \tag{2.12}$$

where $\varepsilon_t \sim iid\,(0, \sigma^2)$. The $NLADF$ statistic for testing the null (2.6) in this set up is now given by

$$NLADF = \frac{\widehat{\delta}}{s.e.(\widehat{\delta})}, \tag{2.13}$$

where $\widehat{\delta}$ is the OLS estimate of δ from the following auxiliary regression with the p augmentations:

$$\Delta y_t = \sum_{j=1}^{p} \phi_j \Delta y_{t-j} + \delta y_{t-1}^3 + error, \tag{2.14}$$

and $s.e.(\widehat{\delta})$ is its standard error of $\widehat{\delta}$.

Theorem 2.3 *Consider the nonlinear ADF regression (2.12). Under the null (2.6) the NLADF statistic defined by (2.13) has the same asymptotic distribution as obtained under non-serially correlated errors. Furthermore, under the alternative hypothesis, the NLADF statistic is consistent.*

Proof. See the Appendix. ■

[6] Alternatively, we would follow the semi-parametric correction method advanced by Phillips and Perron (1988).

3 Finite Sample Properties

In this section we undertake a small-scale Monte Carlo investigation of the small sample size and power performance of the $NLADF$ test, and compare it with that of the ADF test.

In the first set of experiments we focus mainly on the case where the underlying data process has a non-zero mean but no linear trend. In this case the statistic computed use the data in deviation from mean form. This also ties in nicely with the empirical application to real interest rates (see next section) because real interest rates clearly seem to have non-zero means but are untrended. So we consider the following data generating process for Experiments 1:

$$\Delta y_t = \tau + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \beta_* \left(1 - e^{-\theta y_{t-1}^2}\right) + \varepsilon_t, \tag{3.15}$$

where the error term ε_t is drawn from the standard normal distribution. Notice that τ is zero under the null ($\theta = 0$) and no-zero under the alternative ($\theta > 0$), but without loss of generality we set $\tau = 0$ since the test statistic obtained using the data in its deviation from mean does not depend on non-zero value of τ even under the alternative. Furthermore, in our empirical work, it is found that β_* was very poorly identified and thus not precisely estimated even under the alternative. So for simplicity we set $\beta_* = -1$ throughout all experiments.[7] We also present the results for the case where the series are trended in the second set of Experiments and generated by

$$\Delta y_t = \tau \left(1 + d\left(t - 1\right)\right) + \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} - \left(1 - e^{-\theta y_{t-1}^2}\right) + \varepsilon_t, \tag{3.16}$$

where d is set to 0 under the null ($\theta = 0$) and d is set to unity under the alternative $\theta > 0$. A summary of the parameter values considered for Experiments I and II is given below:

	Experiments: Set I									**Experiments: Set II**	
	$\theta = 0$			$\theta > 0$						$\theta = 0$	$\theta > 0$
Parameters	E_{1a}^N	E_{1b}^N	E_{1c}^N	E_{1a}^A	E_{1b}^A	E_{1c}^A	E_{1d}^A	E_{1e}^A	E_{1f}^A	E_2^N	E_2^A
τ	0	0	0	0	0	0	0	0	0	.05	.001
d	-	-	-	-	-	-	-	-	-	0	1
ϕ_1	0	.4	.4	0	0	0	.4	.4	.4	0	0
ϕ_2	0	0	.2	0	0	0	0	0	.2	0	0
θ	0	0	0	.01	.05	.1	.01	.01	.01	0	.05
p	0	1	2	0	0	0	1	2	2	0	1
$\lambda_{\max}$	1	1	1	.96	.89	.83	.91	.91	.83	1	.89

For each of the experiments we have computed the rejection probability of the null hypothesis for sample sizes of $T = 20, 40, ..., 200$. The nominal size of each of the tests is set at

[7] Taylor and Peel (1998) in their work on real exchange rates encountered similar problem, and also set this parameter to minus unity.

0.05 and the number of replications at 10,000. In general, the correct number of augmentation terms are included in the experiments however the results of an investigation into the consequences for power of over-fitting are also given (see Experiment E^A_{1e}).

Table 2: Rejection Frequencies (percent) for Experiment I

		T									
Experiments		20	40	60	80	100	120	140	160	180	200
						size					
E^N_{1a}	*ADF*	10.8	7.4	6.9	6.8	6.3	6.2	5.9	5.8	5.8	5.4
	NLADF	6.0	4.3	4.1	4.2	4.3	4.3	4.5	4.6	4.5	4.9
E^N_{1b}	*ADF*	8.9	6.7	61.	5.7	5.8	5.5	5.4	5.4	5.3	5.3
	NLADF	6.1	4.7	4.8	4.7	4.1	4.6	4.9	4.8	4.8	4.9
E^N_{1c}	*ADF*	9.5	7.1	6.6	6.3	5.8	5.4	5.4	5.2	5.4	5.3
	NLADF	6.2	5.1	5.1	4.8	4.6	4.6	4.5	4.5	4.6	4.5
						power					
E^A_{1a}	*ADF*	12.4	15.0	19.2	25.5	35.4	49.3	66.2	81.1	91.4	97.1
	NLADF	8.8	12.9	20.8	32.6	47.2	63.2	75.6	85.4	91.6	95.2
E^A_{1b}	*ADF*	20.2	37.9	71.5	94.3	99.5	99.9	100	100	100	100
	NLADF	20.3	45.8	74.7	91.4	97.8	99.5	99.9	100	100	100
E^A_{1c}	*ADF*	27.9	69.9	96.6	99.9	100	100	100	100	100	100
	NLADF	32.2	70.7	93.3	98.9	99.9	100	100	100	100	100
E^A_{1d}	*ADF*	17.9	27.5	46.2	68.7	85.7	94.9	98.4	99.7	99.9	100
	NLADF	16.6	33.5	56.6	76.3	89.1	95.5	98.3	99.5	99.8	100
E^A_{1e}	*ADF*	17.9	36.1	62.4	82.6	93.9	98.4	99.6	100	100	100
	NLADF	21.6	44.5	71.4	87.8	95.6	98.6	99.6	100	100	100
E^A_{1f}	*ADF*	9.3	11.1	14.7	19.9	27.1	36.2	47.3	58.8	70.4	80.2
	NLADF	7.7	10.1	16.4	25.6	37.3	50.0	61.3	72.1	81.6	87.9

Turning to size performance of the *ADF* and *NLADF* statistics, we see that none of the tests suffer substantial size distortions, but whereas the *NLADF* tests are typically slightly undersized, the *ADF* test often tend to slightly oversized. Table 2 also gives power estimates under various alternative ($\theta > 0$). In order to get a rough feel for the persistence displayed by each DGP under the alternative, we evaluate the absolute value of the roots of each model at $\Phi\left(E(y^2_{t-1})\right)$, see the values of $\lambda_{\max}$ in the table describing the experiments above. Hence, the largest root displayed indicate the degree of persistence in the region where y^2_{t-1} is close to its mean. Of course following large shocks, y^2_{t-1} will be far from its mean and the degree of local persistence at such points will be less than that indicated by the roots given in the table. The $\lambda_{\max}$ therefore might be interpreted as indicating an "upper bound" on the degree of persistence implied by the associated DGP. The table indicates that for experiments E^A_{1a}, E^A_{1b} and E^A_{1c} the DGP would appear to be the (locally most persistent) because the largest root exceeds 0.8.[8] Given that monthly sample sizes typically exceed 120 observations and

[8]Largest root of 0.9 is frequently found when a linear *AR* model is fitted to monthly time series data such as the real exchange rate and largest roots of .8 are typically found in quarterly real interest rates. See, for example, Driffill and Snell (1998).

quarterly samples lie in the range 80 to 120, the columns tabulating power for $T \geq 80$ are more likely to be of most interest here. In most cases considered here we find a considerable power gain of the $NLADF$ test over the ADF test.[9]

It is well known that if an insufficient number of augmentation terms are included in the ADF regression then the asymptotic distribution of the test will involve nuisance parameters associated with the serial correlation structure of the series in question and results in incorrect (asymptotic) size. To avoid these problems, investigators often err on the side of caution when deciding on the number of augmentation terms and frequently run the risk of "over-fitting" the model. Whilst this may be a sensible precaution against incorrect size, asking the data to estimate too many parameters may have an adverse effect on power. To examine the effect of over-fitting on power for the $NLADF$ test, we have recomputed power for experiment E_{1e}^{A}. The results indicate that whilst over-fitting reduces power in all cases, the $NLADF$ test is clearly less affected than ADF in this respect.

Table 3: Rejection Frequencies (percent) for Experiment II

		T									
Experiments		20	40	60	80	100	120	140	160	180	200
		size									
E_2^N	ADF	11.0	7.7	6.4	6.3	5.9	5.6	5.6	5.5	5.1	5.6
	$NLADF$	6.7	4.7	4.2	3.8	4.0	4.1	4.4	4.7	4.3	4.4
		power									
E_2^A	ADF	17.9	28.6	51.1	77.9	94.0	99.1	99.9	100	100	100
	$NLADF$	16.2	32.0	54.1	77.0	90.1	96.1	99.0	100	100	100

Size and power of the $NLADF$ and ADF tests were computed for a random walk with drift of .01 per time period $\left(E_2^N\right)$ and an $ESTAR(1)$ model with $\theta = .05$ and a linear time trend increment of .001 per time period $\left(E_2^A\right)$. The drift and time trend respectively were chosen to give the series an approximate underlying deterministic trend of .01 per quarter which is the order of magnitude of the trend underlying many quarterly (log) macroeconomic time series. The size and power estimates are tabulated respectively in Table 3. The results on size are more or less in line with the non-trended counterparts, whilst power in all the cells is (understandably) lower when each cell is compared with its non-trended counterpart. The advantage of the $NLADF$ test over the ADF test remains intact, although somewhat smaller than before and only up to a power of 90%.

To further investigate and thus highlight the relative performance of the $NLADF$ and ADF tests, we also consider a third set of experiments in which the DGP is given by

$$y_t = \beta y_{t-1} + \beta_* \left(1 - e^{-\theta y_{t-1}^2}\right) y_{t-1} + \varepsilon_t, \tag{3.17}$$

where $\theta > 0$, $\lambda > 1$, $\beta^* < 0$ and $|\beta + \beta^*| < 1$. This is the special case of the alternative hypotheses in the sense that the process is globally geometrically ergodic and stable, while it

[9] We have also found in other experiments not shown here that the $NLADF$ test outperforms the ADF test for nearly every sample size in less (locally) persistent cases.

is locally explosive.[10] Though this case is not explicitly covered in our theoretical discussion (*i.e.* under the maintained assumption $\beta = 1$), this case is also of clear interest at least in empirics.[11] The parameter values considered for Experiment III are given below (remind that the rest of the specification is common with that of Experiments I and II):

	Experiments: Set III			
Parameters	E^A_{3a}	E^A_{3b}	E^A_{3c}	E^A_{3d}
τ	0	0	0	0
β	1.3	1.3	1.5	1.5
β^*	-0.4	-0.4	-0.6	-0.6
θ	0.05	0.25	0.05	0.25

Table 4 presents the simulation results for the Experiments set III. It is clear that the ADF test is extremely ill behaved when faced with such processes. Invariably, for cases where the locally explosive nature of the process is pronounced because of low values of θ the ADF test has actual power very close to zero. On the other hand the $NLADF$ test is well behaved and correctly concludes that the process is globally stable.

Table 4: Rejection Frequencies (percent) for Experiment III											
		T									
Experiments		20	40	60	80	100	120	140	160	180	200
power											
E^A_{3a}	ADF	2.6	0.3	0.06	0.05	0.07	0.05	0.06	0.1	0.07	0.15
	$NLADF$	0.67	1.3	3.9	12.7	31.3	56.3	78.4	90.8	96.3	98.8
E^A_{3b}	ADF	7.1	6.8	11.0	18.5	28.9	43.5	56.4	67.8	77.5	84.6
	$NLADF$	3.2	11.1	26.7	44.6	61.6	74.0	82.9	88.3	92.6	95.1
E^A_{3c}	ADF	1.0	0.06	0.02	0.0	0.0	0.0	0.01	0.0	0.0	0.0
	$NLADF$	0.2	0.5	1.6	7.4	25.0	53.2	78.5	93.0	98.2	99.5
E^A_{3d}	ADF	4.7	3.3	4.5	7.6	13.9	22.9	34.2	45.2	55.2	64.2
	$NLADF$	2.5	11.3	28.2	48.8	66.7	79.7	87.2	92.9	95.6	97.3

4 Empirical Application to Real interest rates

The apparent unit root behaviour in real interest rates has become an awkward puzzle for economists (see, for example, Barro and Sala-i-Martin (1990) and Rose (1988)). We argued above that transactions costs in financial assets markets is likely to lead to nonlinear speeds of convergence to equilibrium of rates of return. In the context of real interest rates, the Fisher hypothesis predicts that the long run equilibrium value is a common constant. Real rates may temporarily deviate from this constant due to the impact of country specific capital (productivity) shocks but over time adjustments in the capital stock via investment flows

[10]In the corridor regime for small y^2_{t-1}, the process is explosive, but is geometrically ergodic since it has stable dynamics for large y^2_{t-1} . Geometric ergodicity and the associated asymptotic stationarity may be easily established by the drift condition of Tweedie (1975). See also Kurtz and Potter (1991).

[11]In principle, (derivation and application of) the joint test of $\beta = 1$ and $\theta = 0$ might be more desirable, but at a cost of more complexity, which is beyond scope of this study.

should drive the rate back to its common constant mean value. The apparent nonstationarity of real interest rates clearly violates this hypothesis creating the interest rate paradox referred to above. However, the nonstationarity of rates may be an artefact of the ADF tests traditionally used in this context. Owing to transaction costs and other frictions, it is quite plausible that under the Fisher hypothesis, the more that real interests rate deviate from their equilibrium values, the larger will be the investment flows that drive them back again. If so, the results in this paper suggest that the failure to reject the unit root may be due to the lack of power of the standard ADF test and we should use the $NLADF$ procedure. Here we apply the $NLADF$ test to six ex post real interest rates and also estimate the alternative $ESTAR$ models for each.

Quarterly data on ex post 3 month treasury Bill rates for the US, Japan (JA), Germany (GE), France (FR), the UK and Canada (CA) were collected for the 140 quarters covering the period 1959(1) to 1994(4). The price level data in each case is the consumer price index.[12] We treat ex post rates as imperfect measures of their ex ante counterparts. Of course, because the difference between the two is a forecast error, which in most economic worlds is a stationary non-persistent process (white noise under rational expectations), the measurement error involved in using ex post rates to measure their economically meaningful counterparts is not likely to have a profound impact on our inference or estimation results.$\hat{\theta}$

Table 5. Unit root test statistics and ESTAR Estimates for six real interest rates

Country	$ADF2$	$NLADF2$	$\hat{\theta}$	$s.e.(\hat{\theta})$	R^2_{AR}	R^2_{ESTAR}
US	-2.19	-3.50**	.018	.006	.067	.114
JA	-2.88*	-4.43**	.045	.011	.122	.189
GE	-3.23*	-2.95*	.032	.012	.128	.120
FR	-2.61	-3.19*	.022	.008	.142	.161
UK	-2.50	-2.36	.032	.014	.057	.052
CA	-2.14	-3.58**	.018	.006	.087	.135
Note: * (**) denotes significance at 5% (1%) level.						

Most reasonable economic theories would predict that real interest rates are trendless with a non-zero mean and graphs of our data confirm this. Significance tests on augmentation terms indicated that one such term was sufficient to account for the correlation in the error term and the results from applying the $NLADF$ and ADF tests to the data are given in Table 5. In all but one case, that of the UK, $NLADF$ is significant at the 5% level and in all but three, (GE, FR and UK) is significant at the 1% level. In marked contrast and in keeping with previous studies cited in the introduction above, the ADF test only rejects a unit root at the 5% level in only two cases, that of JA and GE, and the former of these is very marginal. None of the ADF statistics reject at the 1% level.

Table 5 also displays the results from estimation of the $ESTAR$ model. Initial estimation found β^* to be very poorly identified a result that has been found elsewhere (Taylor and Peel, 1998). As a consequence, we follow the procedure of the numerical section above and set $\beta^*$to minus unity. The table shows that all estimates of θ are correctly signed and although it does not provide a valid significance test in the usual way, the 95% asymptotic confidence

[12]The data we use are taken from the International Financial Statistics.

interval computed under the alternative for θ does not include zero. Finally we should note that in all but two cases, that of GE and UK, adopting the $ESTAR$ alternative in preference to the linear AR model substantially improves fit as measured by (the non mean-corrected) R^2. In the two exceptions of GE and UK, the fit obtained using a linear AR process is only marginally higher than its $ESTAR$ counterpart.

These results suggest that the $NLADF$ test is a useful supplementary statistic to employ in standard unit root testing especially where the series is known to be highly persistent but expected a priori to be stationary. The estimates also suggest that the $ESTAR$ model itself may provide a better alternative to a linear AR model in such cases.

5 Concluding Remarks

Empirical univariate analysis of nonstationarity against stationarity has been an integral part of time series econometrics prior to the multivariate analysis. However, the emphasis of the earlier literature was on the examination of the linear model, implicitly disregarding any possible nonlinearities remaining in the series under consideration. Recently, some analyses have attempted to fill this vacuum. This paper follows this line of studies, and develops a nonlinear unit root test statistic designed to be more powerful against the stationary exponential transition autoregressive series. This paper demonstrates an advance over the existing literature in three senses. *First,* we derive the limiting nonstandard distribution of the proposed $NLADF$ tests. *Second*, we establish the superior power performance of the $NLADF$ test over the standard linear ADF test under the alternative of nonlinear stationarity via Monte Carlo simulation exercises. *Third*, we provide an application to *ex post* real interest rates from six major OECD countries, and find the $NLADF$ test is able to reject a unit root in cases, whereas the linear ADF tests fail.

A Appendix

A.1 Proof of Theorem 2.1

The asymptotic null distribution of the $NLADF$ statistic defined by (2.9) can be derived as follows. First, under the null (2.6),

$$NLADF = \frac{\sum_{t=1}^{T} y_{t-1}^3 \varepsilon_t}{\sqrt{\hat{\sigma}^2 \sum_{t=1}^{T} y_{t-1}^6}},$$

where $\hat{\sigma}^2$ is the least squares estimate of σ^2 from the auxiliary regression. It is easy to show that $\hat{\sigma}^2 \to_p \sigma^2$ under the null, so we only need to find the asymptotics for $\sum_{t=1}^{T} y_{t-1}^3 \varepsilon_t$ and $\sum_{t=1}^{T} y_{t-1}^6$. For the former, it is easy to show that

$$\frac{1}{T^4} \sum_{t=1}^{T} y_{t-1}^6 \Rightarrow \sigma^6 \int W(r)^6 dr,$$

whereas for the latter we employ the identity

$$y_t^4 = (y_{t-1} + \varepsilon_t)^4 = y_{t-1}^4 + 4y_{t-1}^3 \varepsilon_t + 6y_{t-1}^2 \varepsilon_t^2 + 4y_{t-1}\varepsilon_t^3 + \varepsilon_t^4,$$

so that we have

$$y_{t-1}^3 \varepsilon_t = \frac{1}{4} \left(y_t^4 - y_{t-1}^4 - 6y_{t-1}^2 \varepsilon_t^2 - 4y_{t-1}\varepsilon_t^3 - \varepsilon_t^4 \right).$$

Summing the above expression over $t = 1, ..., T$, and assuming $y_0 = 0$, then we obtain

$$\sum_{t=1}^{T} y_{t-1}^3 \varepsilon_t = \frac{1}{4} y_T^4 - \frac{3}{2} \sum_{t=1}^{T} y_{t-1}^2 \varepsilon_t^2 - \sum_{t=1}^{T} y_{t-1}\varepsilon_t^3 - \frac{1}{4} \sum_{t=1}^{T} \varepsilon_t^4.$$

From standard results (*e.g.* Chan and Wei (1988)) we have

$$\left(\frac{1}{\sqrt{T}} y_T \right)^4 \Rightarrow \sigma^4 W(1)^4,$$

$$\frac{1}{T^2} \sum_{t=1}^{T} \varepsilon_t^4 = O_p\left(T^{-1}\right),$$

But we need to show that

$$\frac{1}{T^2} \sum_{t=1}^{T} y_{t-1}^2 \varepsilon_t^2 \Rightarrow \sigma^4 \int_0^1 W^2(r)\, dr,$$

$$\frac{1}{T^2} \sum_{t=1}^{T} y_{t-1} \varepsilon_t^3 = O_p\left(T^{-\frac{1}{2}}\right).$$

First, write

$$\frac{\sum_{t=1}^{T} y_{t-1}^2 \varepsilon_t^2}{T^2} = \frac{\sum_{t=1}^{T} y_{t-1}^2 (\varepsilon_t^2 - \sigma^2)}{T^2} + \sigma^2 \frac{\sum_{t=1}^{T} y_{t-1}^2}{T^2} = A + B.$$

Then we show that A is $O_p(T^{-1})$. Note that the *iid* assumption for ε_t implies that the elements summed in A are mean zero and have zero covariance, *i.e.*

$$E\left(y_{t-1}^2(\varepsilon_t^2 - \sigma^2)\right) = E(y_{t-1}^2)E(\varepsilon_t^2 - \sigma^2) = 0 \; \forall \; t,$$

and

$$E\left\{y_{j-1}^2(\varepsilon_j^2 - \sigma^2) y_{k-1}^2(\varepsilon_k^2 - \sigma^2)\right\} = E\left\{\left(y_{j-1}^2(\varepsilon_j^2 - \sigma^2) E y_{k-1}^2\right)\right\} E(\varepsilon_k^2 - \sigma^2) = 0, \quad \forall \; k > j.$$

Squaring A, taking expectations and using the above results gives

$$\begin{aligned} Var\,(A) &= E(A^2) = \frac{\sum_{t=1}^{T} E\left\{y_{t-1}^4(\varepsilon_t^2 - \sigma^2)^2\right\}}{T^4} \\ &= \frac{\sum_{t=1}^{T} E(y_{t-1}^4) E(\varepsilon_t^2 - \sigma^2)^2}{T^4} = \frac{E\left\{(\varepsilon_t^2 - \sigma^2)^2\right\} \sum_{t=1}^{T} E(y_{t-1}^4)}{T^4}. \end{aligned}$$

We have assumed that $E(\varepsilon_t^r) = k_r < \infty$ for $r \leq 6$, and $E(\varepsilon_t^2 - \sigma^2)^2 = c < \infty$. Using this in the last term of the above equation gives

$$Var\,(A) = \frac{c}{T^4} \sum_{t=1}^{T} [t k_4 + t(t-1) k_2] = O(T^{-1}).$$

So by Chebyshev's inequality, A is $O_p(T^{-1})$, which proves that

$$\frac{1}{T^2} \sum_{t=1}^{T} y_{t-1}^2 \varepsilon_t^2 \Rightarrow \sigma^4 \int_0^1 W^2\,(r)\, dr.$$

as required. An almost identical argument may be used to show that $\frac{\sum_{t=1}^{T} y_{t-1} \varepsilon_t^3}{T^2}$ is also $O_p(T^{-1})$. Hence,

$$\frac{1}{T^2} \sum_{t=1}^{T} y_{t-1}^3 \varepsilon_t \Rightarrow \sigma^4 \left\{\frac{1}{4} W\,(1)^4 - \frac{3}{2} \int_0^1 W\,(r)^2\, dr\right\},$$

and (2.11) follows. ■

A.2 Proof of Theorem 2.2

Under the alternative (2.7), $\Delta y_t, y_{t-1}$ and y_{t-1}^3 are $I(0)$ and it is easy to show that

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1}^3 \Delta y_t = O_p\,(1)\,; \; \frac{1}{T} \sum_{t=1}^{T} y_{t-1}^6 = O_p\,(1)\,.$$

Therefore,

$$\frac{\sum_{t=1}^{T} \Delta y_t y_{t-1}^3}{\sqrt{\hat{\sigma}^2 \sum_{t=1}^{T} y_{t-1}^6}} = O_p\left(T^{\frac{1}{2}}\right).$$

Hence, the *NLADF* statistic diverges to infinity at rate $T^{\frac{1}{2}}$ under the alternative. ■

A.3 Proof of Theorem 2.3

Define the $T \times p$ data matrix $\mathbf{Z} = (\Delta \mathbf{y}_{-1}, ..., \Delta \mathbf{y}_{-p})$ with $\Delta \mathbf{y}_{-i} = (\Delta y_{-i+1}, ..., \Delta y_{T-i})$, and the $T \times T$ idempotent matrix $\mathbf{M}_T = \mathbf{I}_T - \mathbf{Z} (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{Z}'$. Now,

$$\widehat{\sigma^2} = \frac{1}{T}\boldsymbol{\varepsilon}'\mathbf{M}_T\boldsymbol{\varepsilon} = \frac{1}{T}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} + o_p(1) \xrightarrow{p} \sigma^2,$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_T)'$. Furthermore, under the null it is straightforward to show that

$$\frac{1}{T^2}\mathbf{y}_{-1}^{3\prime}\mathbf{M}_T\boldsymbol{\varepsilon} = \frac{1}{T^2}\mathbf{y}_{-1}^{3\prime}\boldsymbol{\varepsilon}_t + o_p(1) \Rightarrow \sigma^4 \left\{ \frac{1}{4}W(1)^4 - \frac{3}{2}\int_0^1 W(r)^2\, dr \right\},$$

$$\frac{1}{T^4}\mathbf{y}_{-1}^{3\prime}\mathbf{M}_T\mathbf{y}_{-1}^3 = \frac{1}{T^4}\mathbf{y}_{-1}^{3\prime}\mathbf{y}_{-1}^3 + o_p(1) \Rightarrow \sigma^6 \int W(r)^6 dr.$$

where $\mathbf{y}_{-1}^3 = \left(y_0^3, y_1^3, ..., y_{T-1}^3\right)'$. Using these results, under the null we obtain

$$NLADF = \frac{\frac{1}{T^2}\mathbf{y}_{-1}^{3\prime}\mathbf{M}_T\boldsymbol{\varepsilon}}{\sqrt{\widehat{\sigma^2}\frac{1}{T^4}\mathbf{y}_{-1}^{3\prime}\mathbf{M}_T\mathbf{y}_{-1}^3}} = \frac{\frac{1}{T^2}\mathbf{y}_{-1}^{3\prime}\boldsymbol{\varepsilon}}{\frac{1}{T^4}\mathbf{y}_{-1}^{3\prime}\mathbf{y}_{-1}^3} + o_p(1),$$

which as we have shown before has the asymptotic distribution given in (2.11).

Along similar lines in the proof of Theorem 2.2, it is easily seen that the $NLADF$ test is consistent under (2.7). ■

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