where I is the identity matrix. Or more generally the Jacobian of any $\overline{BR}$ dynamic at any mixed equilibrium point q', irrespective of whether $q' = q$, from (5) and Lemma 2 is given by

$$J_{\overline{BR}} = \frac{1}{\gamma} Q_B A - I. \tag{9}$$

If we consider the behavior of the dynamics when the perturbation becomes very small, that is when $\gamma \to 0$, then the following is clear—we approach the *PDA* dynamics.

PROPOSITION 1. *Consider a fully mixed strategy equilibrium q derived from a game matrix A. If A is negative or positive definite with respect to $\mathbb{R}^n_0$, there exists a γ sufficiently small such that a perturbed equilibrium q' has the same stability properties under $\overline{BR}$ dynamics as q has under PDA dynamics. If A is constant sum then q is a center for the linearized PDA dynamics, and q' is asymptotically stable for $\overline{BR}$ dynamics for any $\gamma > 0$.*

Proof. The linearization of any *PDA* dynamic of form (6) at a fully mixed equilibrium is simply given by QA. Given that A is negative (positive) definite with respect to $\mathbb{R}^n_0$ then from Lemma 1 all $n-1$ eigenvalues of QA when constrained to $\mathbb{R}^n_0$ are negative (positive). It follows from Lemma 2 that similarly the relevant eigenvalues of $Q_B A$ evaluated at q' must all be negative (positive). If μ is such an eigenvalue of $Q_B A$ then there is a corresponding eigenvalue of $J_{\overline{BR}}$ equal to $\mu/\gamma - 1$. Clearly for some γ sufficiently close to 0 the eigenvalues of QA and $J_{\overline{BR}}$ have the same sign pattern.

In the case of a constant sum game, $z \cdot Az = 0$, $\forall z \in \mathbb{R}^n_0$. This has the consequence that the eigenvalues of QA, if Q is a *PDA* operator, constrained to $\mathbb{R}^n_0$ have zero real part. Hence the eigenvalues of the linearization of the $\overline{BR}$ dynamics are of the form $-1 \pm i\,\alpha/\gamma$, where α is a constant, possibly zero. Hence, the equilibrium is asymptotically stable. ■

We have seen that when the level of disturbance γ is very close to zero, $\overline{BR}$ dynamics give the same results as *PDA* dynamics except in the zero sum case. It is natural to conjecture that, when the level of perturbation reaches zero, and we have the *BR* dynamics, the same result applies. This is indeed the case as was shown by Hofbauer (1995). However, we can now better understand why there is such a strong relationship between *PDA* and *BR* dynamics.[4] I give part of Hofbauer's results and I sketch his proof. Clearly as the *BR* dynamic is not differentiable at q, a linearization cannot be

[4]Gaunersdorfer and Hofbauer (995) also show a close rela ionship be ween *BR* dynamics and he evolu ionary replica or dynamics (which is *PDA*) even when hey bo h diverge from Nash equilibrium

constructed. Rather Hofbauer designs an appropriate Liapunov function. At any time, the BR dynamic moves toward the best response, which, if strategy i currently has the highest payoff, will be the vertex of the simplex where $x_i = 1$. However, as we noted at the end of the previous section, if a game matrix is positive definite then an increase in x_i will increase $(Ax)_i$. Thus, the system can move a long way away from equilibrium before i ceases to be the best response. A converse argument holds when A is negative definite.

PROPOSITION 2. *If a game matrix A is negative definite with respect to $\mathbb{R}_0^n$ or constant sum, any fully mixed equilibrium q is asymptotically stable under BR dynamics. If A is positive definite then q is unstable.*

Proof. This is a sketch of the proof due to Hofbauer (1995) for the special case where $BR(x)$, for any $x \neq q$, is one of the vertices of S_n. We label the vertices e_i for $i = 1, \dots, n$. Normalize $e_i \cdot Ae_i = 0$ for all i. If $j = \arg\max_i (Ax)_i$, then use $V(x) = \max_i (Ax)_i = e_j \cdot Ax$ as a Liapunov function. This will have a minimum at q. Given our normalization, $\dot{V} = -V$. If A is negative definite or zero sum, $V(x) > 0$ for $x \neq q$ and thus $\dot{V} = -V < 0$ and q is asymptotically stable. However, if A is positive definite, given our normalization, $q \cdot Aq < 0$ and thus close to q, $\dot{V} = -V > 0$. ■

To understand these results better, consider the following game,

$$A = \begin{array}{|c|c|c|} \hline 0 & a & -b \\ \hline -b & 0 & a \\ \hline a & -b & 0 \\ \hline \end{array} \qquad a, b > 0. \tag{10}$$

This version of the "Rock-Scissors-Paper" game has a unique mixed equilibrium at $x = (1/3, 1/3, 1/3)$. It can be shown that if $a > b$ then A is negative definite with respect to $\mathbb{R}_0^n$ and that if $a < b$ then it is positive definite. While if $a = b$, the game is zero sum. Now we can see that in the first case, the mixed equilibrium is stable for all three dynamics PDA, BR, and $\overline{BR}$. In the second case it is unstable for the PDA, BR and, if the level of noise γ is sufficiently low, $\overline{BR}$ dynamics. In the zero sum case, the BR and $\overline{BR}$ dynamics will converge to the equilibrium. But for PDA dynamics the equilibrium is nonhyperbolic, or more specifically all eigenvalues of the linearization have zero real part, and it becomes impossible to obtain a general result on the behavior of the whole class of PDA dynamics. However, it is well known that for the evolutionary replicator dynamics, see, for example, Hofbauer and Sigmund (1988, p. 130), that the mixed equilibrium is neutrally stable.

There are similar considerations for asymmetric games. We can extend PDA dynamics to $S_n \times S_m$ simply by writing

$$\dot{x} = Q(x)Ay, \qquad \dot{y} = Q(y)Bx,$$

where $Q(x)$ and $Q(y)$ satisfy the conditions outlined in the previous section. We can extend the $\overline{BR}$ dynamics in a similar manner with,

$$\dot{x} = \overline{BR}(y) - x, \qquad \dot{y} = \overline{BR}(x) - y,$$

then the Jacobian taken at a perturbed equilibrium q' will be

$$\begin{pmatrix} 0 & \dfrac{d\overline{BR}(y)}{dy} \\ \dfrac{d\overline{BR}(x)}{dx} & 0 \end{pmatrix} - I = \frac{1}{\gamma}\begin{pmatrix} Q_B(x) & 0 \\ 0 & Q_B(y) \end{pmatrix}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} - I,$$

where $Q_B(x)$ and $Q_B(y)$ are both *PDA* dynamic operators at q'. Again as in the symmetric case, if μ is an eigenvalue for a *PDA* dynamic when linearized around a mixed equilibrium q, then we can find a $\overline{BR}$ dynamic linearized at q' with an eigenvalue $\mu/\gamma - 1$. A simple application is the "Matching Pennies" type game considered by Fudenberg and Kreps (1993) among many others. Matching pennies is a 2×2 zero sum game with a unique (mixed) equilibrium. Just as for the zero sum version of Rock-Scissors-Paper game (i.e., when $a = b$, see above), it is a center for the *PDA* dynamics (the linearization has purely imaginary eigenvalues). It follows that the equilibrium will be asymptotically stable under $\overline{BR}$ dynamics. The point is that, in both symmetric and asymmetric games, the effect of noise is to push the path of the system inward from the boundary of the state space. Under this influence, the closed orbits surrounding a neutrally stable equilibrium point start to spiral inward and the equilibrium becomes asymptotically stable.

We can also relate the results here with those recently produced by Cressman (1997) on monotonic dynamics. The principal condition that a dynamic must satisfy to be called monotonic is that, in our present notation, $\dot{x}_i/x_i > \dot{x}_j/x_j$ iff $(Ax)_i > (Ax)_j$. That is, the proportional growth rate of strategy i is greater than that of strategy j if and only if its current payoff is higher. Now, neither *BR* nor $\overline{BR}$ dynamics meet this condition. This is because for best response dynamics the growth rate of a strategy $\dot{x}_i$ is independent of its population share x_i. Nonetheless, we can make a connection. Cressman shows that the linearization of any smooth (i.e., differentiable) monotonic dynamic is a positive transformation of any hyperbolic linearization of the replicator dynamics. Or in other words the linearization of a monotonic dynamic at an interior mixed strategy equilibrium could be written QA, where Q is a *PDA* dynamic operator. Thus, *PDA* dynamics seem to be the linear form of both monotonic and best response dynamics.

5. CONCLUSION

There are two potential ways to conclude this article. One is to emphasize the differences between different formulations of evolutionary and learning dynamics and one is to emphasize their similarities. For example, if we concentrate our attention on the Rock-Scissors-Paper game (10) when $a = b$, or on asymmetric 2×2 games with a unique mixed equilibrium, then we have the following conflicting results. The mixed equilibrium is asymptotically stable for the $\overline{BR}$ and BR dynamics, and indeed for classical fictitious play. It may be asymptotically stable for some PDA dynamics, and unstable for others, it is certainly neutrally stable for some. Alternatively one can focus one's attention on the case where for (10) $a > b$ and there is complete unanimity on the stability of the unique equilibrium. It does not matter whether one uses BR, $\overline{BR}$, or PDA dynamics.

But even in the former case, it is not so easy to conclude that $\overline{BR}$ and PDA dynamics are clearly differentiated. Particularly, I would resist the argument that best response dynamics are somehow superior because they converge in some cases where evolutionary type dynamics do not. In a recent article, Erev and Roth (1997) examine 2×2 games with a unique mixed strategy equilibrium under reinforcement learning, a type of learning much more naive than fictitious play. Their simulations sometimes show convergence to equilibrium. This is despite earlier theoretical work which has shown that the expected motion of such a stochastic learning model is the same as the replicator dynamic and that learning does not converge to the mixed strategy equilibrium in 2×2 games (Posch, 1997; Börgers and Sarin, 1997). Erev and Roth have, however, modified the basic reinforcement learning model by adding the idea that players experiment. This has the effect that the expected motion of the stochastic process is given by the replicator dynamics plus an additional stabilizing component. In fact, it is very similar in form to the linearization of $\overline{BR}$ dynamic that we have constructed in this article. However, the exact connection will have to be established by further research.

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