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# Unemployment and econometric learning\*

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## Abstract

We apply well-known results of the econometric learning literature to the Mortensen-Pissarides real business cycle model. The unique rational expectations equilibrium (REE) is always expectationally stable with decreasing gain learning, and this result is robust to over-parametrisation of the econometric model relative to the minimum state variable form used by agents. And so, from this perspective, the assumption of rational expectations in the model is not unreasonable. However, using a parametrisation with UK data, simulations suggest that the implied rate of convergence to the REE with least squares learning is slow. The cyclical response of unemployment to structural shocks is muted under learning, and a parametrisation which guarantees root-t convergence is generally not consistent with attempts to match the observed volatility of labour market data using the standard model.

Key words: real business cycle, unemployment, adaptive learning, expectational stability

JEL Codes: E24, E32, J64

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## 1. Introduction

The Mortensen & Pissarides (1994) model of search frictions has become the foundation for the investigation of the functioning of labour markets (see Rogerson & Shimer, 2011 for a survey). The existence of search frictions in labour markets is usually motivated by the observation that the market is decentralised due to geographical and sectoral differences in firms, and because each worker has distinct features which make them more or less suitable for these jobs. That is, there is fundamental heterogeneity present in labour markets. Whilst searching for a match, unemployed workers and firms must form expectations of future variables relevant to their decision making process, such as the unemployment rate and the quantity of open vacancies. The decentralised nature of labour markets makes it a priori *not* obvious that workers and firms are able to correctly quantify the values of these relevant variables at all times. Nevertheless, the assumption that firms and unemployed workers have rational expectations is usually made in search and matching models. This is likely to pose overly strong requirements on the cognitive abilities of economic agents, and is unrealistic in the presence of potentially frequent structural and policy shocks. Even small departures from the rational expectation assumption might alter qualitative or quantitative predictions, and these should be explored.

Here we analyse the equilibrium properties and dynamics of the textbook real business cycle (RBC) Mortensen-Pissarides model (see Hagedorn & Manovskii, 2008 for an equivalent discrete time treatment) whilst representing agents as ‘good econometricians,’ who form forecasts according to their estimates of structural model parameters (Evans & Honkapohja, 2001). We follow the literature on econometric learning and assume that agents employ a recursive least squares algorithm to update their parameter estimates each period when new data becomes available. Econometric learning can provide a behavioural foundation for the assumption of rational expectations if the rational expectations equilibrium (REE) is shown to be learnable or E-stable (Evans & Honkapohja, 2001); i.e. small deviations from this equilibrium are reversed over time, and the economy and agents’ parameter estimates converge asymptotically to the true equilibrium values. We derive conditions under which the model’s REE is E-stable. No parameter restrictions are required beyond those made in the standard model when agents use a minimum state variable rule to form and update their estimates of the level of so-called labour market tightness. Furthermore, we confirm that the model is in fact globally stable and satisfies the properties of Strong E-stability (i.e. robust to over-parametrisation of the econometric relationship by the agents). And so from this perspective, the assumption of rational expectations when studying or applying this model is not unreasonable.

There exists a significant literature on the representation of the good econometrician in macroeconomic models. Mankiw et al. (2004) offer evidence against the rational expectations hypothesis. Their analysis of surveys of professional forecasters and households containing expected inflation in one year’s time shows significant autocorrelation in the forecast errors, which is compatible with econometric learning, but not with rational expectations. Milani (2007, 2011) also argues for the presence of adaptive learning in DSGE models, using Bayesian methods to estimate a New Keynesian model that nests learning with habit formation, indexation and other ‘mechanical’ ways of generating persistence.

His results show that learning is capable of replacing the other sources of persistence whilst at the same time increasing the fit to the data as compared to rational expectations. Pfajfar & Santoro (2010) examine a survey of households' inflation forecasts during 1978-2005 and conclude that the hypothesis of the presence of rational expectations can be rejected, and that there is evidence in support of adaptive learning dynamics. This view is further supported by Berardi & Galimberti (2012), who examine post-WWII data of US inflation and output growth. Comparing the performance of different adaptive learning algorithms in matching survey forecasts, their results suggest that economic agents form these according to recursive least squares.

Our formulation of learning here is such that agents need only make one-step-ahead forecasts of the labour market tightness using their estimated model parameters and observed productivity. This is not the only recent paper to apply the principles of econometric learning to this class of model. From a similar model starting point, di Pace et al. (2014) consider an alternate formulation in which agents must make infinite horizon forecasts about the future path of wages, unemployment and profits. This type of learning fits into the anticipated utility approach (Kreps, 1998), and has notably been applied to the RBC model by Eusepi & Preston (2011). di Pace et al. (2014) focus on results with constant gain learning, for which there are no equivalent analytical results to the E-stability conditions we consider. The authors use the model to address the so-called 'unemployment volatility puzzle' (Shimer, 2005). Under infinite horizon, they not only match US forecast errors for unemployment, but also generate significant amplification relative to the baseline model. This is driven by persistence or inertia in agents' expectations of the future path of wages, which implies firms are over-optimistic about future profits, post more vacancies, and thus unemployment is more volatile relative to the REE baseline case. They also find some, but significantly less, propagation of unemployment when they reformulate the model in a one-step-ahead forecast guise.

We present illustrative simulations and analyse the dynamics of the unemployment model with econometric learning, and show that the REE could be a poor approximation to an economy in which agents play the role of good econometricians in terms of levels. Convergence to the REE is slow when we give agents only fifty quarters of historical data, and even after we allow for weighted least squares learning. We also show that structural shocks, when agents are learning but initially begin at the REE, generate a more gradual 'cyclical' adjustment of wages, and thus unemployment volatility would be relatively reduced than under rational expectations. Therefore, we find conflicting results to di Pace et al. (2014). Given that we derive a minimum state variable form of the model, in which agents need only estimate the relationship between labour market tightness and productivity to form their expectations and close the model, the 'wage curve' is already built-in. Econometric learning generates inertia in expectations of the labour market state and wages following shocks.<sup>1</sup> And so the same amplification mechanism described in di Pace et al. (2014) is not present. Given the only true choice variable in the model is the number of vacancies that firms open or close each period, our application of learning appears to be a

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<sup>1</sup>Although we do not expand on this point later, it is straightforward to see that the minimum state variable solution of the model we apply, which agents learn and use to form expectations, could be re-written in terms of wages and productivity by substituting for the standard 'wage curve' which derives from the Nash Bargaining solution.

natural way to extend the baseline model.

The equilibrium of the textbook search and match RBC model can be characterised by a level of labour market tightness and the productivity state, which are independent of the state of unemployment. This is reflected in the minimum state variable rule agents could use under econometric learning, whereby they choose and form expectations of tightness. In doing so, we are implicitly assuming that they fully understand the out of steady-state dynamics of unemployment and vacancy creation. The concept of labour market tightness is a theoretical construct of the model, and not something we would expect firms and workers to be forecasting in reality. Thus we consider an alternative learning rule which relaxes this latter implied assumption by setting agents the task of learning how to set vacancy creation. This alternative rule is not E-stable for the complete range of possible model parameters. And where it does converge, it does so more slowly than under the minimum state variable representation. The approximation of the REE would in this case appear to be further weakened.

We also consider in more detail the speed of convergence to the REE of the model. The same parameters and magnitudes which reduce the speed of convergence substantially are the same as those which in the REE model also increase the model's ability to match unemployment volatility: a relatively high flow value of unemployment, low worker bargaining power and low matching function elasticity with respect to unemployment (see Hagedorn & Manovskii (2008) for a detailed discussion of these). To match the observed volatility of unemployment etc., more extreme values of these parameters would be required when we relax rational expectations in this limited way.

The rest of the paper is structured as follows: in section 2 we add learning to the textbook search and match model of unemployment and derive key expectational stability results, in section 3 we analyse and discuss a simple parametrisation of the model and results of stochastic recursive simulations with learning relative to the REE, and in section 4 we summarise.

## **2. The Model**

We briefly outline a standard textbook search and match model of the labour market (Pissarides, 2000) and the resultant discrete time dynamic equation for the state of the labour market, measured by its so-called tightness, thus analogous to the treatment in Hagedorn & Manovskii (2008).

### ***2.1. The labour market***

There is a continuum of identical, risk neutral workers with total measure one, and an infinite horizon. The matching function  $M(u_t, v_t)$  gives the number of successful matches in a given period. It is increasing and concave in both of its arguments  $u_t$  and  $v_t$ , which represent the share of the total workforce currently unemployed and the level of vacancies relative to the size of the workforce respectively. Matches and separations occur after agents in the economy have made decisions, at the end of period  $t$ . A Cobb-Douglas, constant returns to scale matching function is chosen due to its simplicity, well-known

features, and being commonplace in the literature;

$$M(u_t, v_t) = \mu u_t^\alpha v_t^{1-\alpha}, \quad \mu > 0 \quad \alpha \in (0, 1). \quad (1)$$

For the following exposition we define the level of labour market tightness, as

$$\theta_t = \frac{v_t}{u_t}. \quad (2)$$

The probability of a firm filling an open vacancy is

$$q(\theta_t) = \frac{M(u_t, v_t)}{v_t} = \mu \theta_t^{-\alpha}, \quad (3)$$

and the corresponding probability that an unemployed worker gets matched to an open vacancy is

$$\theta_t q(\theta_t) = \frac{M(u_t, v_t)}{u_t} = \mu \theta_t^{1-\alpha}. \quad (4)$$

The law of motion for the share of workers who are unemployed at the beginning of period  $t + 1$  is

$$u_{t+1} = u_t + (1 - u_t)\lambda - \theta_t q(\theta_t)u_t. \quad (5)$$

## 2.2. The household

For ex-positional simplicity we consider an economy comprising a single representative household of size one, in which all workers are identical, risk neutral and there is perfect consumption insurance across members. The household can therefore be represented by the Bellman equation

$$W_t(n, y) = w_t n_t + D_t + b(1 - n_t) + \delta W_{t+1}^e(n, y), \quad (6)$$

with the law of motion for employment given by

$$n_{t+1} = (1 - \lambda)n_t + \theta_t q(\theta_t)(1 - n_t), \quad (7)$$

where  $W_t(n, y)$  represents its current value function, labour productivity is denoted by  $y_t$ , and  $e$  shows time  $t$  subjective expectations. The household takes as given wages  $w_t$ , dividends from the representative firm  $D_t$  and labour market tightness  $\theta_t$ . The period utility value from non-employment is given by  $b$ , and  $\delta$  is the discount factor. Applying the envelope theorem, the surplus from an additional member of the household being employed is given by

$$\frac{\partial W_t(n, y)}{\partial n_t} = w_t - b + \delta(1 - \lambda - \theta_t q(\theta_t)) \frac{\partial W_{t+1}^e(n, y)}{\partial n_{t+1}}, \quad (8)$$

i.e. the net value of employment plus the expected continuation value.

### 2.3. The firm

A representative firm with linear production function maximises profits by choosing the quantity of vacancies to post in each period, at constant ongoing cost  $c$ , subject to the law of motion for the labour market, and taking  $w_t$  and  $\theta_t$  as given. This problem can be represented as the Bellman equation

$$\Pi_t(v; , n, y) = \max_{v_t \geq 0} y_t n_t - w_t n_t - c v_t + \delta \Pi_{t+1}^e(v; , n, y), \quad (9)$$

with the constraint

$$n_{t+1} = (1 - \lambda)n_t + q(\theta_t)v_t, \quad (10)$$

where  $\Pi_t(v; , n, y)$  represents its current value function. The first order condition is given by

$$\frac{\partial \Pi_{t+1}^e(v; n, y)}{\partial n_{t+1}} = \frac{c}{\delta q(\theta_t)}. \quad (11)$$

Applying the envelope theorem, and using the first order condition gives the surplus to the firm from employing an additional worker;

$$\frac{\partial \Pi_t(v; n, y)}{\partial n_t} = y_t - w_t + \frac{(1 - \lambda)c}{q(\theta_t)}, \quad (12)$$

i.e. the net profit from an additional employed worker plus the expected continuation value.

### 2.4. Wage determination

Wages are determined by generalised Nash bargaining between firm and workers over the additional surpluses (8) & (12), with worker bargaining power  $\beta \in (0, 1)$ .<sup>2</sup> Combining the surplus sharing rules which form the solution of this problem, iterating forwards, and using (8) (12) & (11) gives what is referred to in the textbook model as the ‘wage curve’;

$$w_t = (1 - \beta)b + \beta(y_t + c\theta_t). \quad (13)$$

### 2.5. REE and dynamics

The equilibrium of the model can be characterised and determined uniquely by the value of labour market tightness at which point the representative firm is indifferent between opening an additional vacancy or not, due to the assumption of free entry. The non-linear equation determining this value of  $\theta_t$  can be obtained by iterating forwards and taking expectations of the envelope theorem condition with respect to employment for the firm, and then using the firm’s first order condition (11) and the wage curve equation (13);

$$\frac{c}{\delta q(\theta_t)} = (1 - \beta)(y_{t+1}^e - b) + \frac{(1 - \lambda)c}{q(\theta_{t+1}^e)} - \theta_{t+1}^e \beta c. \quad (14)$$

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<sup>2</sup>Note, it is crucial here that both workers and firms are assumed to form expectations in the same way, using the same rule, otherwise the Nash Bargaining solution would be significantly complicated.

To provide intuition for this expression, we stress the similarities to (11). Aggregate labour market tightness  $\theta_t$  will adjust immediately to deviations from this equality via the representative firm instantaneously opening or closing vacancies. Thus, today's labour market tightness is determined by expectations of the value of a filled vacancy in the next period.

We assume that the process of worker productivity takes the form of a stationary AR(1) in logs;

$$\log(y_t) = \rho \log(y_{t-1}) + \varepsilon_t, \quad \rho \in [0, 1) \quad (15)$$

with  $\varepsilon_t$  being an i.i.d. normal shock with variance  $\sigma_\varepsilon^2 > 0$ .

Once agents have formed expectations for  $\theta_{t+1}$  and  $y_{t+1}$ , the remaining endogenous variables of interest such as wages and unemployment follow mechanically using the model equations derived above. To solve the system consisting of (14) and (15) we linearise around steady state values  $\bar{\theta}$  and  $\bar{y} = 1$  (see appendix A.1 for derivation);

$$\theta_t = \psi_0 + \psi_1 y_{t+1}^e + \psi_2 \theta_{t+1}^e, \quad (16)$$

$$y_t = (1 - \rho) + \rho y_{t-1} + \varepsilon_t. \quad (17)$$

In the following, the operator  $E_t x$  denotes mathematical expectations formed at period  $t$  regarding variable  $x$ . The linearised dynamics of output (17) can be substituted into (16) for  $y_{t+1}^e$  by noting that under rational expectations  $y_{t+1}^e = E_t y_{t+1} = (1 - \rho) + \rho y_t$ ;

$$\theta_t = \tilde{\psi}_0 + \tilde{\psi}_1 y_{t-1} + \tilde{\psi}_2 \theta_{t+1}^e + \tilde{\psi}_1 \rho^{-1} \varepsilon_t, \quad (18)$$

with

$$\tilde{\psi}_0 = \psi_0 + \psi_1 (1 - \rho)(1 + \rho),$$

$$\tilde{\psi}_1 = \psi_1 \rho^2,$$

$$\tilde{\psi}_2 = \psi_2.$$

A REE of the system (17) and (18) is a stochastic process  $\theta_t$  that satisfies this system with  $E_t \theta_{t+1} = \theta_{t+1}^e$ . As is well known, linear RE models in which agents form expectations regarding an endogenous variable can possess multiple equilibria (or *bubble* solutions, i.e. which are not related to economic fundamentals).<sup>3</sup> If there are multiple *stable* rational expectations equilibria then a model is said to be indeterminate. To see the possibility of this, note that (18) can be written in ARMA(1,1) form as

$$\theta_t = \tilde{\psi}_2^{-1} (\rho^{-1} \tilde{\psi}_1 (1 - \rho) - \tilde{\psi}_0) - \tilde{\psi}_1 \tilde{\psi}_2^{-1} \rho^{-1} y_{t-1} + \tilde{\psi}_2^{-1} \theta_{t-1} + d_1 \varepsilon_t + d_2 \eta_t + \tilde{\psi}_1 \rho^{-1} (\tilde{\psi}_2^{-1} - 1) \varepsilon_{t-1}, \quad (19)$$

<sup>3</sup>The literature on bubbles and the related concept of indeterminacy is reviewed in Benhabib & Farmer (1999). The classic reference on bubble solutions is Blanchard & Watson (1983); see also Bullard & Mitra (2002) for a recent analysis of indeterminacy in a New Keynesian framework.



with  $d_1$  and  $d_2$  arbitrary parameters, and  $\eta_t := E_{t+1}[\theta_{t+2}] - E_t[\theta_{t+2}]$  being a martingale difference sequence such that  $E_t[\eta_{t+1}] = 0$ . No restrictions are imposed on  $d_1$  or  $d_2$  since rational expectations formed according to (19) always result in an identity regardless of these parameters. Therefore, there is a continuum of possible solutions to (19). Evans & Honkapohja (1986) have shown that any finite degree ARMA solution of an equation of the form (18) can at most be ARMA(1,1) and the particular form of (19) nests all possible ARMA solutions of finite degree. However, the ARMA class of solutions is stable if  $|\tilde{\psi}_2| > 1$  and is unstable for  $|\tilde{\psi}_2| < 1$ , in which case the unique stationary solution to (17) and (18) is the fundamental or minimal state variable (MSV) solution. The MSV solution depends on a minimal set of state variables such that it is impossible to delete any of the variables from the set and still obtain solutions to (18) and (15) for all permitted parameter values (McCallum, 1983). The MSV solution can be guessed to be of the form

$$\theta_t = A + By_{t-1} + C\varepsilon_t, \quad (20)$$

since the only predetermined variable in the structural equation (18) is  $y_{t-1}$ . In the appendix A.2 it is shown how (20) can be obtained from (19). We first focus on this solution since it is the natural benchmark for learning and the stability of equilibria (McCallum, 1983; Evans & Honkapohja, 2001). If agents are not able to learn the simplest representation (as few state variables as possible), they cannot be expected to be able to learn equilibria containing more state variables and to coordinate their behaviour towards them.

## 2.6. E-Stability of the MSV Solution

We now relax the assumption of rational expectations by modelling agents as econometricians attempting to estimate the parameters underpinning the true motion of the economy under uncertainty. Therefore, agents are endowed with a perceived law of motion (PLM) of the economy of the MSV form (20). The task for agents is then to estimate the values of the parameters  $A$ ,  $B$  and  $C$ . They form estimates in period  $t$  given by  $\hat{A}_t$ ,  $\hat{B}_t$  and  $\hat{C}_t$ , and update these estimates every period when new data becomes available. We assume agents perform this updating and estimation task using the recursive least squares estimator. This is the most widely used estimation technique in the learning literature and Berardi & Galimberti (2012) provide evidence that this estimator matches the observed survey forecasts of US time series closely.<sup>4</sup> According to the PLM, the forecast of next period's labour market tightness is

$$\theta_{t+1}^e = \hat{A}_t + \hat{B}_t((1 - \rho) + \rho y_{t-1} + \varepsilon_t). \quad (21)$$

There are potential problems of simultaneity in forward looking models. Therefore, it is assumed that although agents forecast  $\theta_{t+1}$  by using  $y_t$ , the variable  $y_t$  is *not* in the information set for the estimation of  $\hat{A}_t$  and  $\hat{B}_t$ . However, as proved by Marcet & Sargent (1989), this does not alter the asymptotic stability results obtained in the following as compared to an algorithm allowing for simultaneity so long as agents are assumed to ignore outliers, defined as being an observation outside some predetermined range. Inserting the econometric forecast into (18) gives the actual law of motion (ALM) for labour

<sup>4</sup>The presented algorithm is comparable to a restricted form of the well-known Kalman filter. For further discussion of this issue see Berardi & Galimberti (2013).

market tightness;

$$\theta_t = \tilde{\psi}_0 + \tilde{\psi}_2 \hat{A}_t + \tilde{\psi}_2 (1 - \rho) \hat{B}_t + (\tilde{\psi}_1 + \tilde{\psi}_2 \rho \hat{B}_t) y_{t-1} + (\tilde{\psi}_2 \hat{B}_t + \tilde{\psi}_1 \rho^{-1}) \varepsilon_t. \quad (22)$$

This defines the following  $T$ -mapping from the PLM to the ALM

$$T(\hat{A}_t, \hat{B}_t, \hat{C}_t) = (\tilde{\psi}_0 + \tilde{\psi}_2 \hat{A}_t + \tilde{\psi}_2 (1 - \rho) \hat{B}_t, \tilde{\psi}_1 + \tilde{\psi}_2 \rho \hat{B}_t, \tilde{\psi}_2 \hat{B}_t + \tilde{\psi}_1 \rho^{-1}). \quad (23)$$

It is apparent that the  $T$ -mapping to  $\hat{C}_t$  is determined by the other coefficients, and thus the estimate  $\hat{C}_t$  is independent of  $C$ , and does not influence the stability results. Therefore, in what follows we refer to the mappings  $T(\hat{A}_t, \hat{B}_t)$ , and for  $\hat{C}_t$ ,  $V(\hat{B}_t)$ . There is a self-referential feature inherent in all learning models which can be seen in equation (22). Although the parameters of their estimation are non-stationary during the transition to steady state values, agents neglect this fact whilst forming their estimates since a least squares method assumes the ‘true’  $A$ ,  $B$  and  $C$  to be constants. Intuitively, if the coefficient which determines the responsiveness to expectations is sufficiently small, this specification error becomes asymptotically negligible and the economy converges to the REE (Evans & Honkapohja, 2001).

The estimation takes place via recursive least squares (RLS). Let  $z'_{t-1} = (1, y_{t-1})$ ,  $x'_{t-1} = (\hat{A}_t, \hat{B}_t)$  and

$$\theta_t = z'_{t-1} x_{t-1} + \eta_t. \quad (24)$$

The estimation error  $\eta_t$  is perceived by the agents to be *iid*. However, due to the self-referential nature of the model there is an endogeneity bias which agents are unaware of and thus  $\eta_t$  is *not* truly *iid*. We now define  $S_t = \sum_{i=1}^t z_{i-1} z'_{i-1}$  which allows us to write the RLS estimator as

$$S_t = S_{t-1} + z_{t-1} z'_{t-1}, \quad (25)$$

$$x_t = x_{t-1} + S_{t-1}^{-1} z_{t-1} (\theta_t - z'_{t-1} x_{t-1}). \quad (26)$$

Further, with  $R_t \equiv t^{-1} S_t$ ,

$$R_t = R_{t-1} + t^{-1} (z_{t-1} z'_{t-1} - R_{t-1}), \quad (27)$$

$$x_t = x_{t-1} + t^{-1} R_t^{-1} z_{t-1} (\theta_t - z'_{t-1} x_{t-1}), \quad (28)$$

and thus

$$x_t = x_{t-1} + t^{-1} R_t^{-1} z_{t-1} \{ z'_{t-1} [T(\hat{A}_t, \hat{B}_t) - x_{t-1}] + V(\hat{B}_t) \varepsilon_t \}, \quad (29)$$

with the gain sequence  $1/t$ , often referred to as decreasing gain learning.<sup>5</sup> This gain guarantees that asymptotically new information is disregarded by agents.

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<sup>5</sup>In the case of constant gain learning the weight given each observation is geometrically declining with the time since it was observed, and the gain sequence would be  $0 < \gamma < 1$ .

For the rational expectations equilibrium to be asymptotically stable under learning, the E-stability principle has to be fulfilled. According to this the stability of the system in (27) and (29) with decreasing gain is governed by the following ordinary differential equation (ODE), where  $\tau$  denotes ‘notional’ time:

$$\frac{d}{d\tau} (\hat{A}, \hat{B}) = T(\hat{A}, \hat{B}) - (\hat{A}, \hat{B}). \quad (30)$$

The REE is then the unique fixed point of these differential equations. With this specification, it can be shown that the REE of (18) is stable under learning, i.e. small deviations from the steady state value  $\bar{\theta}$  are reversed, if the sufficient condition,  $\tilde{\psi}_2 < 1$ , holds. The necessary condition for E-stability is  $\tilde{\psi}_2 \rho < 1$ . In appendix A.4 we show that the sufficient condition always holds for this model, and we have global convergence to the REE (see A.5). As was explained in the previous section, the model is determinate if  $|\tilde{\psi}_2| < 1$ . We therefore can state the following:

**Proposition 2.1.** *If the economy described by the system (17) and (18) exhibits determinacy and the PLM is of the MSV form, and if agents learn using least squares updating, then so long as  $\tilde{\psi}_2 < 1$  the unique REE is E-stable.*

**Corollary 2.2.** *The textbook equilibrium model of labour market search and match frictions, with homogeneous agents and no on the job search (Pissarides, 2000: Chapter 1), is E-stable.*

## 2.7. Strong E-stability of the MSV Solution

One potential criticism of the econometric learning literature is that it is not necessarily clear how agents could settle upon a particular law of motion for the economy. Strong E-stability of a system is defined if the previous result is robust to over-parametrisation of the PLM (Evans & Honkapohja, 2001). Assume that agents are forming their expectations of  $\theta_{t+1}$  according to the general ARMA representation (19), and are not endowed with a PLM of the MSV form. Moreover, due to econometric considerations they start with an arbitrarily over-parametrised version,

$$\theta_t = a + \sum_{j=1}^s b_j y_{t-j} + \sum_{j=1}^r c_j \theta_{t-j} + \sum_{j=1}^q d_j \varepsilon_{t-j} + \sum_{j=1}^l f_j \eta_{t-j} + d_0 \varepsilon_t + f_0 \eta_t. \quad (31)$$

Accordingly, expectations of  $\theta_{t+1}$  take the form

$$\theta_{t+1}^e = a + \sum_{j=1}^s b_j y_{t+1-j} + \sum_{j=1}^r c_j \theta_{t+1-j} + \sum_{j=1}^q d_j \varepsilon_{t+1-j} + \sum_{j=1}^l f_j \eta_{t+1-j}, \quad (32)$$

which can be substituted into equation (18) to obtain the new ALM and a corresponding  $T$ -mapping in exactly the same way as before (see appendix A.6). Let  $\mathbf{b}' = (b_1, \dots, b_s)$ ,  $\mathbf{c}' = (c_1, \dots, c_r)$ ,  $\mathbf{d}' = (d_0, \dots, d_q)$ , and also  $\mathbf{f}' = (f_0, \dots, f_l)$ . Further, define  $\phi' = (a, \mathbf{b}', \mathbf{c}', \mathbf{d}', \mathbf{f}')$ . According to the E-stability principle, the ODE governing the stability of the above system is given by

$$\frac{d\phi}{d\tau} = T(\phi) - \phi. \quad (33)$$

To investigate whether agents will detect the over-parametrisation and converge towards the MSV solution, the stability of (33) at the REE must be studied. In appendix A.6 we show the following:

**Proposition 2.3.** *If the economy described by the system (17) and (18) exhibits determinacy and the PLM is of the over-parametrised ARMA form, and if agents learn using least squares updating, then so long as  $\tilde{\psi}_2 < 1$  the unique REE is E-stable.*

**Corollary 2.4.** *The textbook equilibrium model of labour market search and match frictions, with homogeneous agents and no on the job search (Pissarides, 2000: Chapter 1), is Strongly E-stable.*

### 3. Analysis

In the following section we present a brief analysis of the unemployment model with econometric learning described above. We consider two illustrative simulations to demonstrate the implied speed of convergence and dynamics of the endogenous variables in the standard search model with learning. First, we demonstrate E-stability when starting ‘realistically’ far away from the REE. Second, with agents initially having assumed to have learned the REE, we consider the impact of a shift implied by an arbitrary change in some estimated parameter value. We then discuss the speed of convergence of the model, results under constant gain learning, and the implications if agents use an alternative PLM which does not implicitly assume they understand the out of steady-state dynamics of unemployment and vacancy creation.

#### 3.1. Simulations

Since this is not a strict calibration exercise, we follow a relatively ad hoc and illustrative only parametrisation strategy using seasonally adjusted UK quarterly data for the period 1998-2013.<sup>6</sup>

We normalise average productivity to be one. For the productivity process we estimate an AR(1) in log deviations from trend of output per worker,<sup>7</sup> and find an auto-regressive parameter  $\rho$  for the period of 0.84 and standard deviation of the shocks  $\sigma_\epsilon$  to be 0.0063 (assuming them to be normally distributed). We estimate the aggregate matching function below using OLS as per Sahin et al. (2014) for 2002q1-2013q2.<sup>8</sup> As is common in the literature, we approximate a two state labour market simply by ignoring those who are inactive. We proxy the number of matches, or new hires, as the total inflow into employment  $m_t$  in any quarter, and use the total number of vacancies in the economy.

$$\log\left(\frac{m_t}{u_t}\right) = \log(\mu) + (1 - \alpha)\log\left(\frac{v_t}{u_t}\right) + \zeta_t. \quad (34)$$

<sup>6</sup>All data used from ONS, downloaded 01/08/2014. Labour market data for those aged 16-64 only. For a more complete calibration of the unemployment model using UK data see Burgess & Turon (2010).

<sup>7</sup>Dynamically de-trended using Hodrick & Prescott (1997) filter with standard quarterly smoothing parameter.

<sup>8</sup>Also following Borowczyk-Martins et al. (2013) we consider time trends in the estimation to account for the endogeneity of unobserved shifts in the matching efficiency with the number of vacancies that firms open, but these all drop out. We also carry out tests that the matching function is Cobb-Douglas, and reject the alternative. In line with the existing literature, we find that the data suggests the matching function has decreasing returns to scale, although we proceed as though it is constant; see Petrongolo & Pissarides (2001) for a thorough review of estimates of the aggregate matching function.

We find estimates of  $\alpha = 0.39$  and  $\mu = 1.08$ . For the exogenous separation we use the UK Labour Force Survey two quarter hazard rate estimate for leaving employment, which for 2002q1-2013q2 gives a quarterly average rate of  $\lambda = 0.035$ . Brief summary statistics of the key model quarterly stock and rate variables for the UK consistent with the parametrisation method here, are described in table 1.<sup>9</sup> The discount factor is set as  $\delta = 0.99$ , and to restrict the number of free parameters we let the bargaining power adhere to the Hosios condition,  $\beta = \alpha = 0.39$ . We also set the flow value of unemployment to 0.8.<sup>10</sup> The remaining parameter, the flow vacancy cost  $c$  is chosen to match the observed level of average labour market tightness. The full list of parameters and implied values of the endogenous variables for the deterministic equilibrium are given below in table 2.

Table 1: Level & quarterly rate summary statistics:  
Consistent with model parametrisation, 2002q1-2013q2

	Mean	Std err.
Tightness - $\theta_t = \frac{v_t}{u_t}$	0.35	0.039
Job finding rate - $\theta_t q(\theta_t) = \frac{m_t}{u_t}$	0.59	0.045
Job separation rate - $\lambda_t$	0.035	0.0021
'Equilibrium unemployment' - $u_t^* = \frac{\lambda_t}{\lambda_t + \frac{m_t}{u_t}}$	0.058	0.0088
Actual unemployment rate, 16-64	0.058	0.0039

Source: authors' calculations using UK Labour Force Survey and Labour Market Statistics.

For completeness we write-out in full the stochastic recursive sequence that represents the model, stating from period  $t_0$ :

$$\begin{aligned}
 (I) \quad u_{t+2} &= \lambda(1 - u_{t+1}) + \left[1 - \mu (z_t' T(x_t) + V(x_t) \varepsilon_{t+1})^{1-\alpha}\right] u_{t+1}, \\
 (II) \quad R_{t+1} &= R_t + \frac{1}{t+1} (z_t z_t' - R_t), \\
 (III) \quad x_{t+1} &= x_t + \frac{1}{t+1} R_{t+1}^{-1} z_t \{z_t' [T(x_t) - x_t] + V(x_t) \varepsilon_{t+1}\}, \\
 (IV) \quad y_{t+1} &= (1 - \rho) + \rho y_t + \varepsilon_{t+1}, \\
 (V) \quad \varepsilon_{t+2} &\sim i.i.d. N(0, \sigma^2).
 \end{aligned}$$

When written out in sequence order, the simultaneity which requires us to exclude  $y_t$  from the information set used to estimate  $x_t$  becomes clearer. The adaptive learning process, which takes place at the beginning of each period, can also be represented by figure 1.

<sup>9</sup>All regressed on cubic trend to account for low frequency shifts for the short period in question.

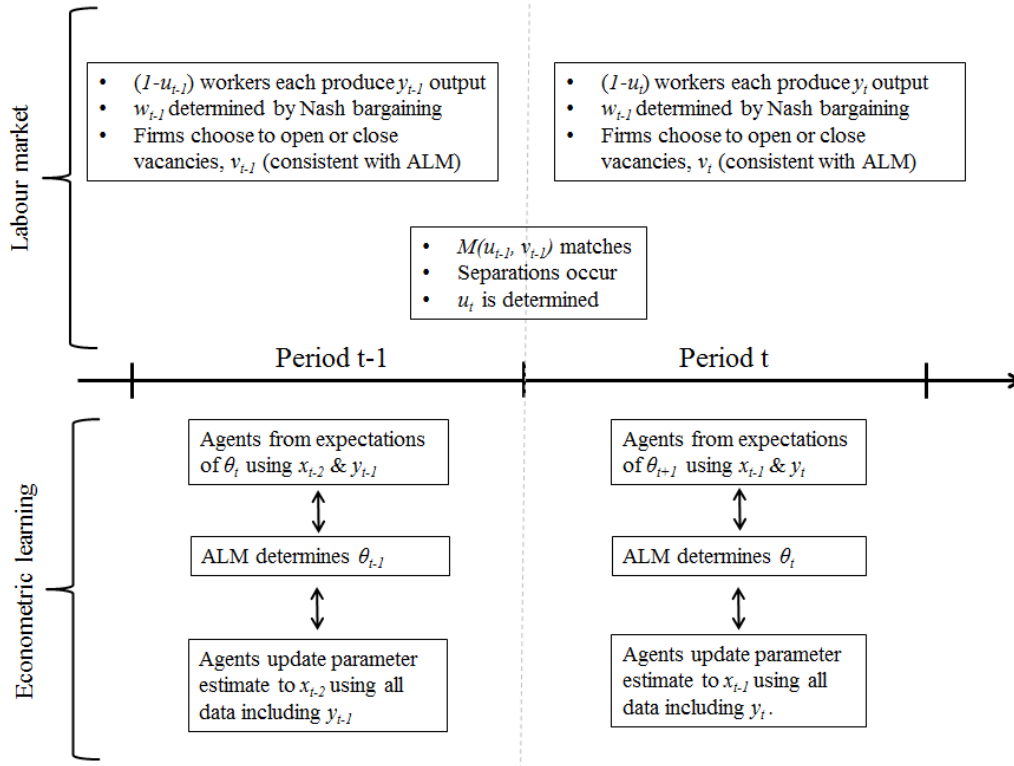
<sup>10</sup>How to select or estimate appropriate values of both the bargaining power and the flow value of unemployment are open to significant academic debate. Shimer (2005) and subsequently Hagedorn & Manovskii (2008) are often considered in the literature as the extreme examples for parametrisations, and highlight how this greatly affects the ability of the model to match the observed volatility of unemployment and vacancy creation. With the arbitrary parametrisation here we are somewhere in between these two extreme values.

Table 2: Assumed/estimated parameter values and deterministic equilibrium

Deterministic parameter	Assumed value
$y$	1
$b$	0.8
$c$	0.74
$\lambda$	0.035
$\mu$	1.08
$\alpha$	0.39
$\beta$	0.39
$\delta$	0.99
$\rho$	0.84
$\sigma$	0.006
Endogenous variable	Deterministic equilibrium
$\theta$	0.35
$u$	0.058
$v$	0.02
$w$	0.98

Source: authors' calculations.

Figure 1: Timeline of the labour market & agents' learning



To initiate the sequence from  $t_0$  we must choose initial values  $u_1$ ,  $x_0$ ,  $z_0$ ,  $S_0$  and  $\varepsilon_1$ . The asymptotic properties of decreasing or constant gains least squares recursion will hold irrespective of the initial conditions. As suggested by Carceles-Poveda & Giannitsarou (2007) the approach taken to set initial values

$z_0$  and  $S_0$  should depend on the particular model in question and the empirical purpose of the researchers. One approach we might take is to use historic or randomly generated data, with  $t_0$  set sufficiently large such that  $S_0$  is invertible; in this case  $t_0 \geq 2$ . This approach is most useful when attempting to compare the performance of models with learning and the assumption of agents as being ‘good econometricians’ against real data. However, this gives few clues as to how large  $t_0$  should be, and the subsequent simulation is likely to be sensitive to this assumed level of memory of agents, particularly for decreasing gain least squares. Another attractive option is to choose initial values from an assumed distribution around the REE.

To set the initial conditions here we use the same data used to parametrise the model to estimate (35): where  $cubtr_t$  represents a cubic time trend to account for the possibility that agents recognise low frequency structural breaks in the relationship, productivity is normalised but not de-trended, and we include significant MA terms to account for otherwise unaccounted for auto-correlation when the MSV is applied to real world data, which the good econometrician may in practice account for by what we have referred to before as an over-parametrised PLM.<sup>11</sup>  $t_0 = 50$  is the maximum number of UK observations available to us.

$$\theta_t = A + cubtr_t + By_{t-1} + \kappa_1 \zeta_{t-1} + \kappa_2 \zeta_{t-2} + \zeta_t, \quad t = 2001q2...2013q3. \quad (35)$$

Using this approach, we set  $S_0 = \begin{pmatrix} 50 & 50 \\ 50 & 50.075 \end{pmatrix}$ ,  $x'_0 = (-1.22, 1.77)$ ,  $z'_0 = (1, 1)$ ,  $\varepsilon_1 = 0$  and

$$u_1 = \frac{\lambda}{\lambda + \mu (z'_0 T(x_0))^{1-\alpha}} \quad (= 0.052).$$

To analyse the impact of adding adaptive learning to the basic search frictions model of the labour market we should focus on the simulated time paths of wages and the tightness parameter, which are independent of  $u_1$ . With the parametrisation as described above the REE parameters of the MSV solution are given by  $x'_{REE} = (-0.9, 1.25)$ . That is to say the elasticity of  $\theta$  to productivity at the long-run average level is around three, which is significantly lower than observed in the data.

Figure 2 (see appendix B) demonstrates a simulation for the baseline case of agents with rational expectations.<sup>12</sup> Unsurprisingly, as is common with this class of models, and as described in table 3, the REE does not come close to matching the key observed moments for labour market tightness and unemployment. Figure 3 shows results from a simulation with decreasing gain learning taking initial values as described above. The key result is that when giving agents a relatively small amount of historical data (12.5 years), and initial estimates of the model parameters which are not unrealistically far from the true REE values, decreasing gain learning convergence is very slow. As shown by figure 4 this takes roughly five hundred years despite being exponential. This indicates that under adaptive learning,

<sup>11</sup>In determining initial conditions, one could also consider the class of GARCH, error correction, or even VAR models, however we believe this would be an unnecessarily significant leap from the straightforward least squares updating we assume that a ‘good econometrician’ carries out in practice, and which constitutes the learning algorithm we study here.

<sup>12</sup>Random number seed set to 42 in Python Numpy application for all simulations.

an economy could be persistently away from its REE level of unemployment, on the high or low side, even though agents are behaving rationally in the limited sense prescribed by the ‘good econometrician.’ In this sense, rational expectations is a poor approximation to a model with learning in terms of levels. One recommendation from this result is that when calibrating the Mortensen-Pissarides model, targeting second moments of the data should always be preferable, whereas not hitting levels of the endogenous variables should not be too concerning.

Table 3: Simulation results

	St.dev $t \leq t_0 + 20$	St.dev $t \leq t_0 + 100$	Min $t \leq t_0 + 100$	Max
Rational expectations eq.				
$w$	0.01	0.008	0.96	0.99
$u$	0.0014	0.0012	0.056	0.061
$\theta$	0.018	0.014	0.032	0.038
Learning (I) - least squares				
$w$	0.011	0.009	0.98	1.02
$u$	0.0014	0.0014	0.05	0.056
$\theta$	0.023	0.019	0.37	0.45
Learning (IV) - constant gain $\gamma = 0.05$				
$w$	0.013	0.010	0.97	1.01
$u$	0.0018	0.0022	0.051	0.059
$\theta$	0.027	0.026	0.33	0.44

Source: authors’ calculations.

As a further example, in figure 5 we simulate the model with no memory, and allow the agents to have guessed the correct initial parameter estimates  $x'_0 = x'_{REE}$ , but suppose that there is an immediate negative ten percent shock to the flow value of unemployment. Under rational expectations, due to the rise in the surplus of a match, firms immediately open more vacancies, and the unemployment rate falls. Under learning, the initial increase in  $\theta$  is smaller. Therefore, unemployment falls more slowly as agents effectively attempt to disentangle the effects of the structural shock from the stochastic process. In this sense, the response to the shock leads to a less volatile response for unemployment. Thus, if actual labour market series feature the effects of frequent permanent shocks of this kind, econometric learning will not improve the ability of the model to match their cyclicity.

### 3.2. Speed of convergence

As shown theoretically in (Benveniste et al., 2012), the learning of the agents results in root-t convergence to the true REE parameter estimates if all the eigenvalues of the Jacobian of the system have real part  $< 1/2$ .<sup>13</sup> As shown, in appendix A.4 when we derive the E-stability conditions, this requires that  $\tilde{\psi} < 1/2$ . In the example parametrisation above (figure 3) this is ensured, with  $\tilde{\psi} = 0.39$ . In fact, more

<sup>13</sup>I.e. the rate at which in classical econometrics the mean of the least squares parameter estimate converges to the true value.



generally, it can also be shown with simulations that the speed of convergence decreases substantially as the value of  $\tilde{\psi}$  moves closer to one, the threshold for E-Stability. To illustrate this, in figure 6 we consider a doubling of  $\tilde{\psi}$  by decreasing worker bargaining power to  $\beta = 0.1$ , keeping all other parameters except  $c$  constant, which we again use to match our target value of  $\theta$ . As expected, the rate of convergence is slowed substantially, and the economy remains more persistently away from the REE. A higher speed of convergence is one way in which the REE model could become an improved approximation of an alternative with econometric learning.

### **3.3. Constant gain learning**

In figure 7, for completeness we also compare the results of our first simulation with decreasing gain learning to an equivalent example with a gain parameter  $\gamma_t = 0.05$ .<sup>14</sup> As expected, when agents weight recent data more highly, convergence to the REE is faster, and agents' parameter estimates are more volatile. This faster convergence also results in more volatile series of labour market tightness, wages and unemployment. However, the gain parameter we use here to guarantee faster convergence implies agents roughly only use data over the past twenty quarters to update their beliefs, and is notably outside the range suggested by the adaptive learning literature (see for example di Pace et al. (2014) for a discussion). The simulation results with constant gain learning, with more reasonable levels of memory weighting, are not dissimilar to those with decreasing gain.

### **3.4. An alternative non-steady-state perceived law of motion**

So far we have described a model of econometric learning in which agents endeavour to forecast labour market tightness  $\theta$ . However, this is a construct of the model and its assumptions. It is an attractive feature of the search and matching models that the equilibrium can be described by this single choice variable, determined by the state of the productivity process, but independent of the level of unemployment. However, firms in the model are in fact described in terms of choosing vacancies, or whether or not to enter the labour market. And for given levels of  $\theta$  and productivity, this choice does depend on the state of the labour market. In fact, in the rational expectations equilibrium, if we consider a situation where the model is at its steady state (i.e. the unemployment and vacancy rates are on the Beveridge curve), then following some one-off shock, the level of vacancies is non-monotonic in moving to the new steady state. In characterising agents as learning how to choose and forecast  $\theta$ , we imply that they fully understand the non-steady state dynamics of the model. Here we consider the implications of relaxing this assumption (for what follows, see appendix A.7 for full derivation of results and parameter values).

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<sup>14</sup>For constant gain learning there is no analytic solution for expectational stability and so we must select a reasonably small gain parameters to ensure convergence.

Linearising the system determining the full equilibrium given by (10), (14) & (15) we derive an alternative system defining the economy,

$$v_t = \kappa_0 + \kappa_1 y_{t+1}^e + \kappa_2 v_{t+1}^e + \kappa_3 n_{t+1}^e + \kappa_4 n_t, \quad (36)$$

$$n_t = \phi_0 + \phi_1 n_{t-1} + \phi_2 v_{t-1}, \quad (37)$$

$$y_t = (1 - \rho) + \rho y_{t-1} + \varepsilon_t. \quad (38)$$

We endow agents with a PLM by which they use both the productivity and employment rate states to forecast vacancy creation,

$$v_t = \hat{A}_t + \hat{B}_t y_t + \hat{C}_t n_{t-1}. \quad (39)$$

Given (36)-(39), we can then derive the ALM for this version of the economy, and subsequently a  $\tilde{T}$ -mapping

$$\tilde{T}(\hat{A}_t, \hat{B}_t, \hat{C}_t) = (\tilde{\kappa}_0 + \tilde{\kappa}_2 [\hat{A}_t + (1 - \rho)\hat{B}_t], \tilde{\kappa}_1 + \tilde{\kappa}_2 \rho \hat{B}_t, \tilde{\kappa}_3 + \tilde{\kappa}_2 \hat{C}_t). \quad (40)$$

Assuming agents update their parameter estimates for the PLM using the same form of recursive least squares used previously, and applying the same E-stability principle, it can be shown that the sufficient condition to guarantee local convergence to the REE is given by  $\tilde{\kappa}_2 < 1$ . Comparing this with the condition for stability of the MSV-PLM,  $\tilde{\kappa}_2 > \tilde{\psi}_2$ . Hence, convergence to the REE is slower when agents do not implicitly know the out of steady-state dynamics of employment and vacancy creation. The REE model is then a poorer approximation to an economy with econometric learning. What is more, for a subset of parameter values we cannot claim that the model is E-stable. It is less likely to be E-stable in the circumstance of inefficiently high firm entry, although for the parametrisation we have used above, the model would certainly converge.

#### 4. Summary and further discussion

We take the textbook RBC version of the Mortensen-Pissarides model of search and match frictions for the labour market and show that the unique REE is not only always E-stable, but this result is robust to over-parametrisation of the MSV-PLM used by agents (Strong E-stability) with decreasing gain learning. This PLM is assumed to be in the minimum state variable form, and as such agents only use productivity and estimated model parameters to form expectations of future labour market conditions. These local convergence conditions also extend trivially to global convergence. Hence, from the perspective that agents may in practice apply something that looks like econometric learning, the rational expectations assumption in this class of model is not unreasonable. We use recent UK data to parametrise the model, and show that although the model is E-stable, implied convergence can be very slow. Therefore, the rational expectations model of unemployment could be a poor approximation to one in which agents learn econometrically, particularly in the presence of frequent structural or permanent shocks. What is more, the MSV-PLM implicitly assumes that agents understand the out of steady-state path of employment and vacancy creation in the model. When we consider a version of the PLM which relaxes this assumption, we see that convergence is further slowed, and local E-stability of the model is not guaranteed, making

the approximation of the rational expectations model even weaker.

The so-called ‘Shimer puzzle’ addresses the extent to which the RBC version of the Mortensen-Pissarides model can match key moments of economic time series, in particular the volatility of unemployment, and lack thereof for wages. Hagedorn & Manovskii (2008) discuss at length how the model can be calibrated to match these moments. They focus on the roles of a high flow utility from non-employment ( $b$ ), and low worker bargaining power ( $\beta$ ) and matching elasticity ( $\alpha$ ) to match these volatilities. In our linearised system, the sensitivity of  $\theta$  to productivity shocks is determined by the true REE parameter  $B$ , which becomes large as  $\tilde{\psi}$  goes to 1. It is straightforward to show that the same combination of high  $b$ , low  $\beta$  &  $\alpha$  not only implies higher volatility of  $\theta$  and unemployment, but also slower convergence to the REE in the model with learning.<sup>15</sup> Thus, in a world with least squares learning, more ‘extreme’ values of these parameters would be implied to match the observed cyclicity of unemployment, wages etc.. And so the result of relaxing rational expectations in this limited way is to also move us further from addressing the unemployment volatility puzzle.

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<sup>15</sup>Note, for  $\tilde{\psi}_2$  to be decreasing in  $\alpha$  we require  $\bar{\theta} < \frac{1-\alpha}{\alpha}$ , which will generally be true for reasonable parameter values and long-run equilibrium value of labour market tightness.

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## Appendix A. Methodology

### A.1. Linearisation

We take a first order Taylor approximation around the deterministic steady state values of  $\theta$  and  $y_t$ ,  $\bar{\theta}$  and  $\bar{y} = 1$  respectively, approximating the right and the left hand side of equation (14) which is stated here again for convenience,

$$\frac{c}{\delta q(\theta_t)} = (1 - \beta)(y_{t+1}^e - b) + \frac{(1 - \lambda)c}{q(\theta_{t+1}^e)} - \theta_{t+1}^e \beta c.$$

This results in

$$\begin{aligned} \frac{c}{\delta q(\bar{\theta})} - \frac{cq'(\bar{\theta})}{\delta[q(\bar{\theta})]^2}(\theta_t - \bar{\theta}) &= (1 - \beta)(\bar{y} - b) + (1 - \beta)(y_{t+1}^e - \bar{y}) \\ &\quad - \bar{\theta} \beta c - \beta c(\theta_{t+1}^e - \bar{\theta}) + \frac{(1 - \lambda)c}{q(\bar{\theta})} \\ &\quad - \frac{(1 - \lambda)cq'(\bar{\theta})}{[q(\bar{\theta})]^2}(\theta_{t+1}^e - \bar{\theta}). \end{aligned} \quad (41)$$

By noting that

$$\frac{c}{\delta q(\bar{\theta})} = (1 - \beta)(\bar{y} - b) + \frac{(1 - \lambda)c}{q(\bar{\theta})} - \bar{\theta} \beta c$$

must hold in equilibrium according to (14), this steady state condition can be subtracted from both sides of the approximated equation. Then solving explicitly for  $\theta_t$  and defining the functional form  $q(\bar{\theta}) = \mu \bar{\theta}^{-\alpha}$ , (41) becomes

$$\begin{aligned} \theta_t &= \left\{ 1 + \frac{\beta \delta \mu^2 (\bar{\theta})^{-2\alpha}}{\alpha \mu (\bar{\theta})^{-\alpha-1}} - (1 - \lambda) \delta \right\} \bar{\theta} \\ &\quad - \frac{(1 - \beta) \delta \mu^2 (\bar{\theta})^{-2\alpha}}{c \alpha \mu (\bar{\theta})^{-\alpha-1}} \bar{y} \\ &\quad + \frac{(1 - \beta) \delta \mu^2 (\bar{\theta})^{-2\alpha}}{c \alpha \mu (\bar{\theta})^{-\alpha-1}} y_{t+1}^e \\ &\quad + \left\{ - \frac{\beta \delta \mu^2 (\bar{\theta})^{-2\alpha}}{\alpha \mu (\bar{\theta})^{-\alpha-1}} + (1 - \lambda) \delta \right\} \theta_{t+1}^e, \end{aligned} \quad (42)$$

which can be simplified to the form given in the text, (16), with coefficients

$$\begin{aligned} \psi_0 &= [1 - \psi_2] \bar{\theta} - \psi_1 \bar{y}, \\ \psi_1 &= \frac{(1 - \beta) \delta \bar{\theta} q(\bar{\theta})}{c \alpha}, \\ \psi_2 &= \delta \left[ (1 - \lambda) - \frac{\beta \bar{\theta} q(\bar{\theta})}{\alpha} \right], \end{aligned}$$

and with the steady state value for labour market tightness the solution to

$$(1 - \beta)(\bar{y} - b) - \frac{c(\frac{1-\delta}{\delta} + \lambda)}{q(\bar{\theta})} - \beta c \bar{\theta} = 0.$$

## A.2. ARMA(1,1) and the MSV solution

Derivation of MSV solution: (19) can be re-written as

$$\theta_t = \frac{\rho \tilde{\psi}_0 - \tilde{\psi}_1(1-\rho)}{\rho(1-\tilde{\psi}_2)} + \frac{\tilde{\psi}_1}{\rho(L-\tilde{\psi}_2)}y_{t-1} - \frac{d_1 \tilde{\psi}_2}{(L-\tilde{\psi}_2)}\varepsilon_t - \frac{d_2}{(L-\tilde{\psi}_2)}\eta_t, \quad (43)$$

with  $L$  denoting the lag operator such that  $Lx_t = x_{t-1}$ . The parameters  $d_1$  and  $d_2$  can be chosen arbitrarily. In particular, to obtain the MSV solution  $\theta_t = A + By_{t-1} + C\varepsilon_t$  one must first set  $d_2 = 0$ . (43) can be re-written as:

$$\begin{aligned} \theta_t &= \frac{\rho \tilde{\psi}_0 - \tilde{\psi}_1(1-\rho)}{\rho(1-\tilde{\psi}_2)} - (\rho^{-1}\tilde{\psi}_1 y_{t-1} - d_1 \tilde{\psi}_2 \varepsilon_t) \sum_{i=1}^{\infty} \tilde{\psi}_2^{-i} L^{i-1}. \\ \theta_t &= \frac{\rho \tilde{\psi}_0 - \tilde{\psi}_1(1-\rho)}{\rho(1-\tilde{\psi}_2)} + \rho^{-1}\tilde{\psi}_1 \tilde{\psi}_2^{-1}(1-\rho) \sum_{i=1}^{\infty} \left( \sum_{j=1}^i \rho^{-j} \right) \tilde{\psi}_2^{-i} - \rho^{-1}\tilde{\psi}_1 \tilde{\psi}_2^{-1} y_{t-1} \sum_{i=0}^{\infty} (\rho \tilde{\psi}_2)^{-i} \\ &\quad + \varepsilon_{t-1} (\rho^{-1}\tilde{\psi}_1 \tilde{\psi}_2^{-1} \sum_{i=1}^{\infty} \left( \sum_{j=1}^i \rho^{-j} L^{i-j} \right) \tilde{\psi}_2^{-i} + d_1 \sum_{i=1}^{\infty} \tilde{\psi}_2^{-i} L^{i-1}) + d_1 \varepsilon_t. \end{aligned} \quad (44)$$

Therefore, to derive an MSV solution from a broader the class of ARMA(1,1) solutions, in which no lags of  $\varepsilon_t$  can remain, we therefore see from (44) that

$$d_1 = -\frac{\tilde{\psi}_1}{\rho \tilde{\psi}_2} \left( \frac{1}{\rho \tilde{\psi}_2} + \left( \frac{1}{\rho \tilde{\psi}_2} \right)^2 + \left( \frac{1}{\rho \tilde{\psi}_2} \right)^3 + \dots \right), \quad (45)$$

$$= \frac{\tilde{\psi}_1}{\rho \tilde{\psi}_2 (1 - \rho \tilde{\psi}_2)} \quad \text{if } \tilde{\psi}_2 > \frac{1}{\rho} > 1, \quad (46)$$

which corresponds to the condition for stable ARMA(1,1) solutions. Otherwise, the MSV solution cannot be derived from the class of unstable ARMA(1,1) solutions, and is instead the only stable solution.

## A.3. REE Values

The REE values of the parameters  $A$ ,  $B$ , and  $C$  are found using the method of undetermined coefficients.

$$A = \frac{\tilde{\psi}_0}{1-\tilde{\psi}_2} + \frac{\tilde{\psi}_1 \tilde{\psi}_2 (1-\rho)}{(1-\tilde{\psi}_2)(1-\tilde{\psi}_2 \rho)},$$

$$B = \frac{\tilde{\psi}_1}{1-\tilde{\psi}_2 \rho},$$

$$C = B\rho^{-1},$$

where we have assumed that  $\tilde{\psi}_2 \neq 1$  and  $\tilde{\psi}_2 \rho \neq 1$ .

## A.4. E-Stability

As shown by Evans & Honkapohja (2001) an REE is E-stable if the associated ordinary differential equation given in (30) is asymptotically locally stable under learning. This is the case, if all the eigenvalues of the Jacobian of  $T(\hat{A}, \hat{B}) - (\hat{A}, \hat{B})$  have negative real parts. Thus, we must have

$$\tilde{\psi}_2 \rho - 1 < 0$$

and

$$\tilde{\psi}_2 - 1 < 0,$$

whereby the second condition implies the validity of the first. We need to check when the second condition is true. Writing out the term  $\tilde{\psi}_2$  and rearranging, we see that

$$\delta \left[ (1 - \lambda) - \frac{\beta \bar{\theta} q(\bar{\theta})}{\alpha} \right] < 1 \quad (47)$$

is always true for the widest range of sensible assumed parameter values ( $0 \leq \delta < 1$ ,  $\lambda > 0$ ,  $0 < \beta < 1$ ,  $\mu > 0$ ,  $\alpha > 0$ ; and which all imply  $\bar{\theta} > 0$ ) and hence the REE is E-stable.

### A.5. Global convergence

Given the model discussed here has a unique equilibrium, and satisfies the assumptions of Evans & Honkapohja (1998) that guarantee global convergence, we simply apply their *Theorem 2* to the recursive learning algorithm given by (27) and (29).

For  $R_t$ , using  $E z_t z_t' = M_z$ , where  $M_z$  is some positive definite matrix, taking expectations we have the ODE,

$$\frac{dR}{d\tau} = M_z - S, \quad (48)$$

which is clearly globally asymptotically stable and independent of  $x_t$ .

It is possible that for some  $t$   $R_t$  may not be invertible, though this will happen only a finite number of times with probability 1. We modify the algorithm for  $x_t$  to

$$x_t = x_{t-1} + t^{-1} u(R_{t-1}) z_{t-1} \{ z_{t-1}' [T(\hat{A}_t, \hat{B}_t) - x_{t-1}] + \eta_t \}, \quad (49)$$

where  $u(R)$  is a bounded regular function from the space of  $2 \times 2$  matrices to the subspace of positive definite matrices such that  $u(R) = R^{-1}$  in the neighbourhood of  $M_z$ . Then taking expectations the ODE is given by

$$\frac{dx}{d\tau} = u(s) M_z (T(\hat{A}, \hat{B}) - (A, B))' \quad (50)$$

$$= u(s) M_z (\tilde{\psi}_2 - 1) ((\hat{A}, \hat{B}) - (A, B))'. \quad (51)$$

Given that the other requirements of the theorem are trivially satisfied, then it applies, and this differential equation is clearly globally asymptotically stable for  $\tilde{\psi}_2 < 1$ , and this stability is exponential;  $(\hat{A}, \hat{B}) \rightarrow (A, B)$  globally almost surely.



### A.6. ALM and T-mapping ARMA solution and E-stability

$$\begin{aligned} \theta_t = & \frac{\tilde{\psi}_0 + \tilde{\psi}_2(a + b_1(1 - \rho))}{1 - \tilde{\psi}_2c_1} + \frac{\tilde{\psi}_1 + \tilde{\psi}_2(b_2 + b_1\rho)}{1 - \tilde{\psi}_2c_1}y_{t-1} + \frac{\tilde{\psi}_2(b_1 + d_1) + \tilde{\psi}_1\rho^{-1}}{1 - \tilde{\psi}_2c_1}\varepsilon_t + \frac{\tilde{\psi}_2f_1}{1 - \tilde{\psi}_2c_1}\eta_t \\ & + \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} \sum_{j=3}^s b_j y_{t+1-j} + \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} \sum_{j=2}^r c_j \theta_{t+1-j} \\ & + \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} \sum_{j=2}^q d_j \varepsilon_{t+1-j} + \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} \sum_{j=2}^l f_j \eta_{t+1-j}. \end{aligned} \quad (52)$$

This defines again a T-mapping from the PLM to the ALM with corresponding elements:

$$a = \frac{\tilde{\psi}_0 + \tilde{\psi}_2(a + b_1(1 - \rho))}{1 - \tilde{\psi}_2c_1}, \quad (53)$$

$$b_1 = \frac{\tilde{\psi}_1 + \tilde{\psi}_2(b_1\rho + b_2)}{1 - \tilde{\psi}_2c_1}, \quad (54)$$

$$d_0 = \frac{\tilde{\psi}_1\rho^{-1} + \tilde{\psi}_2(b_1 + d_1)}{1 - \tilde{\psi}_2c_1}, \quad (55)$$

$$b_j = \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} b_{j+1}, \quad j = 2, \dots, s-1, \quad b_s = 0, \quad (56)$$

$$c_j = \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} c_{j+1}, \quad j = 1, \dots, r-1, \quad c_r = 0, \quad (57)$$

$$d_j = \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} d_{j+1}, \quad j = 1, \dots, q-1, \quad d_q = 0, \quad (58)$$

$$f_j = \frac{\tilde{\psi}_2}{1 - \tilde{\psi}_2c_1} f_{j+1}, \quad j = 0, \dots, l-1, \quad f_l = 0. \quad (59)$$

Since (53) - (59) describes a non-linear system of differential equations, we first have to linearise (33) to study stability properties. However, the subsystem (57) is independent of the other equations and can be analysed separately. The eigenvalues of the Jacobian of  $T(\mathbf{c}) - \mathbf{c}$  at the REE values  $c_j = 0$  for  $j = 1t, \dots, r$  are found to be  $r$  times repeatedly equal to  $-1$  and therefore the subsystem (57) will converge towards the REE values. Due to the convergence of  $\mathbf{c}$  it is apparent that  $\mathbf{d}$  (apart from  $d_0$ ) and  $\mathbf{f}$  will also converge to their REE values of vectors of zeros. Moreover,  $b_j = 0$  for  $j = 2, \dots, s$  is easily verified to be the values towards which the economy under learning converges. Finally, convergence of  $a$ ,  $b_1$  and  $d_0$  are studied by analysing the Jacobian of the system (53)-(55). If this Jacobian has eigenvalues strictly less than unity, then the whole system is E-stable. It can easily be verified that the eigenvalues are  $\tilde{\psi}_2$  and  $\tilde{\psi}_2\rho$ .

### A.7. A non-steady-state PLM

The system defined as (36)-(38), linearised around steady-state values  $\bar{v}, \bar{n}, \bar{y} = 1$  has derived parameter values as follows,

$$\begin{aligned}\kappa_0 &= (1 - \kappa_2) \bar{\theta} - \bar{y} \kappa_1, \\ \kappa_1 &= \frac{\delta(1 - \beta)(1 - \bar{n}) \bar{\theta} q(\bar{\theta})}{c\alpha}, \\ \kappa_2 &= \delta \left[ (1 - \lambda) - \frac{\beta \bar{\theta} q(\bar{\theta})}{\alpha} \right] \quad (= \psi_2), \\ \kappa_3 &= \bar{\theta} \kappa_2, \\ \kappa_4 &= -\bar{\theta}, \\ \phi_0 &= \alpha q(\bar{\theta})(\bar{v} + \bar{\theta} \bar{n}), \\ \phi_1 &= (1 - \lambda) - \alpha \theta q(\bar{\theta}), \\ \phi_2 &= q(\bar{\theta})(1 - \alpha).\end{aligned}$$

Given the PLM (39), agents form expectations according to

$$v_{t+1}^e = \hat{A}_t + \hat{B}_t [(1 - \rho) + \rho y_t] y_t + \hat{C}_t n_t, \quad (60)$$

and the ALM is given by

$$v_t = \tilde{\kappa}_0 + \tilde{\kappa}_2 [\hat{A}_t + (1 - \rho) \hat{B}_t] + [\tilde{\kappa}_1 + \tilde{\kappa}_2 \rho \hat{B}_t] y_t + [\tilde{\kappa}_3 + \tilde{\kappa}_2 \hat{C}_t] n_t, \quad (61)$$

where

$$\begin{aligned}\tilde{\kappa}_0 &= \frac{\kappa_0 + \kappa_1(1 - \rho) + \kappa_3 \phi_0}{1 - \kappa_3 \phi_2}, \\ \tilde{\kappa}_1 &= \frac{\kappa_1 \rho}{1 - \kappa_3 \phi_2}, \\ \tilde{\kappa}_2 &= \frac{\kappa_2}{1 - \kappa_3 \phi_2}, \\ \tilde{\kappa}_3 &= \frac{\kappa_4 + \kappa_3 \phi_1}{1 - \kappa_3 \phi_2}.\end{aligned}$$

Given the mapping  $\tilde{T}$  defined in the main text, the REE is E-stable if all the eigenvalues of the Jacobian of  $\tilde{T}(\hat{A}, \hat{B}, \hat{C}) - (\hat{A}, \hat{B}, \hat{C})$  have negative real parts. Thus, we must have

$$\tilde{\kappa}_2 \rho - 1 < 0$$

and

$$\tilde{\kappa}_2 - 1 < 0,$$

whereby the second condition implies the validity of the first. Therefore, we need to check what range of parameter values for the model the second condition is true. Writing out the term  $\tilde{\kappa}_2$  and rearranging,

we see that the required condition is

$$\delta \left[ (1 - \lambda) - \frac{\beta \bar{\theta} q(\bar{\theta})}{\alpha} \right] [1 + \bar{\theta} q(\bar{\theta})(1 - \alpha)] < 1, \quad (62)$$

or

$$\tilde{\psi}_2 [1 + \bar{\theta} q(\bar{\theta})(1 - \alpha)] < 1. \quad (63)$$

Given that  $\tilde{\kappa}_2 > \tilde{\psi}_2$ , if the E-stability condition holds with this alternative PLM, convergence will be slower. For the complete range of possible model parameters, this condition does not hold. As realistic levels of  $\lambda$  are small, the condition would be sensitive to assumed parameter values of  $\beta$  and  $\alpha$ . For example, given  $\alpha > \beta = 0$ , which is the case of zero worker bargaining power, whereby wages are reduced to the value of the outside option, and there is excessive firm entry, or inefficiently high according to the Hosios condition, it is more likely E-stability will not hold.

## Appendix B. Simulation Figures

Figure 2: Baseline – Rational expectations eq.

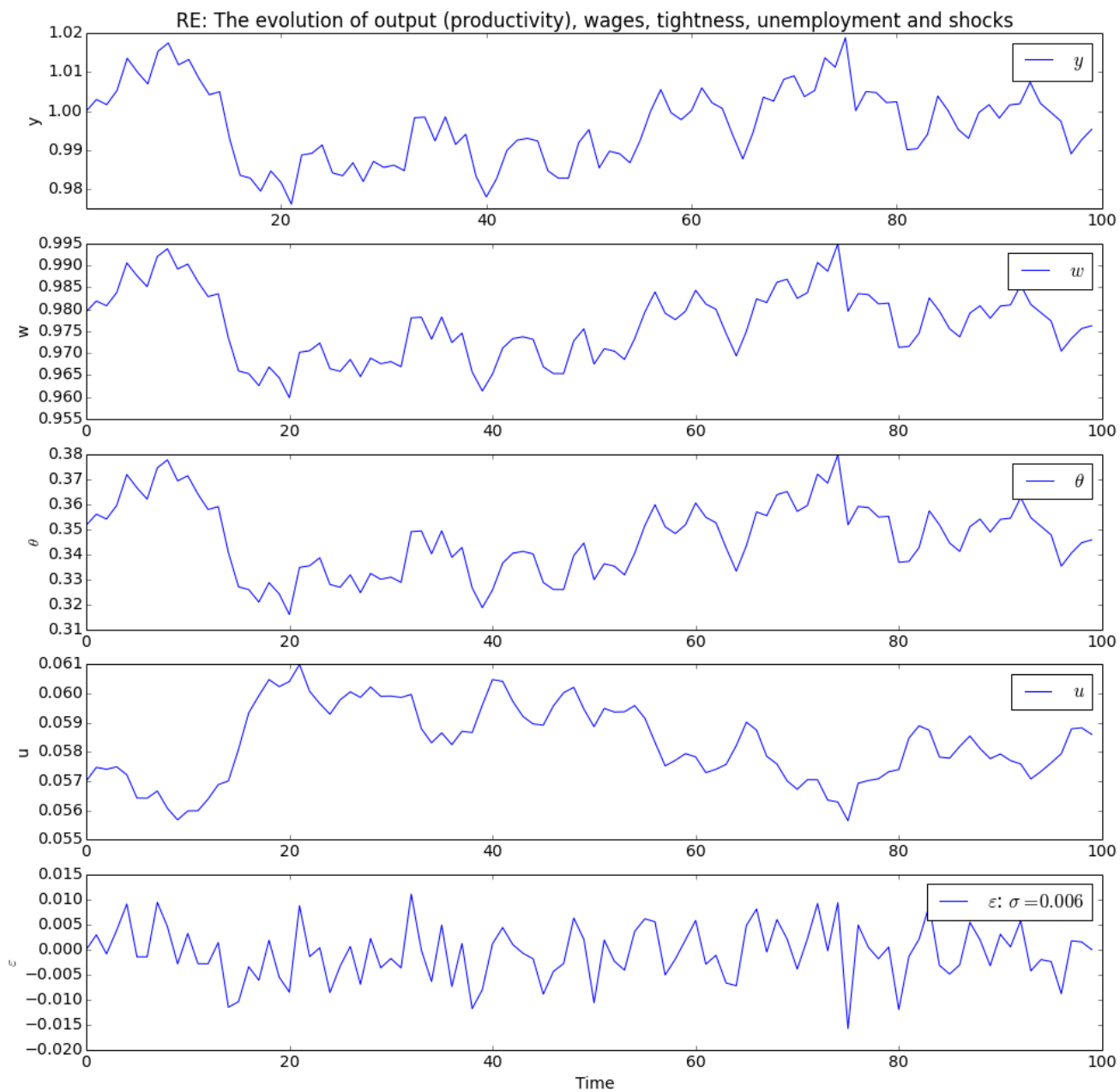
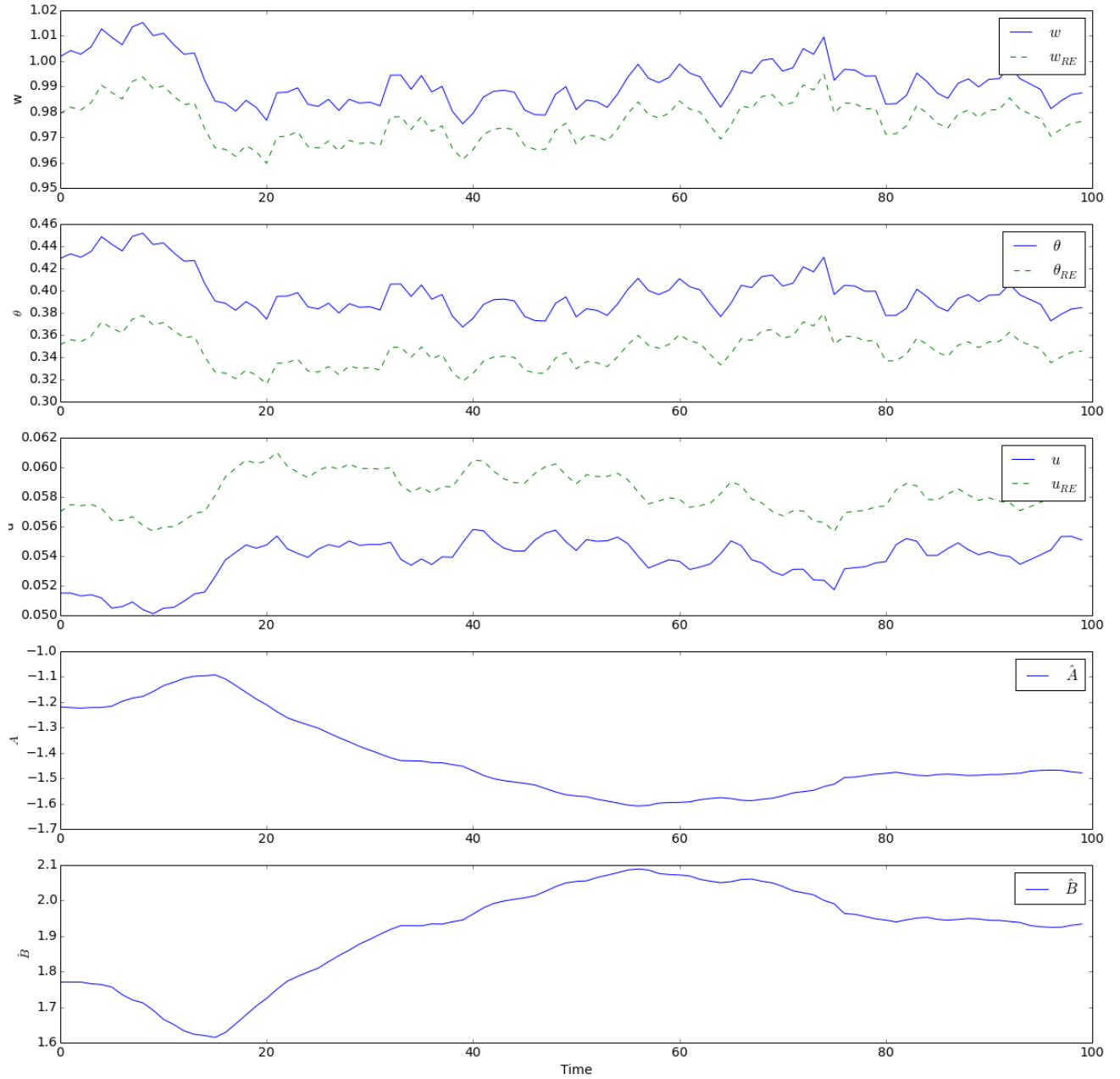


Figure 3: Learning (I) - Comparison of recursive least squares learning with rational expectations eq.



Note.- With initial parameter estimates assumed to be 'realistically' far away from true REE values.

Figure 4: Learning (I) - Convergence of parameter estimates

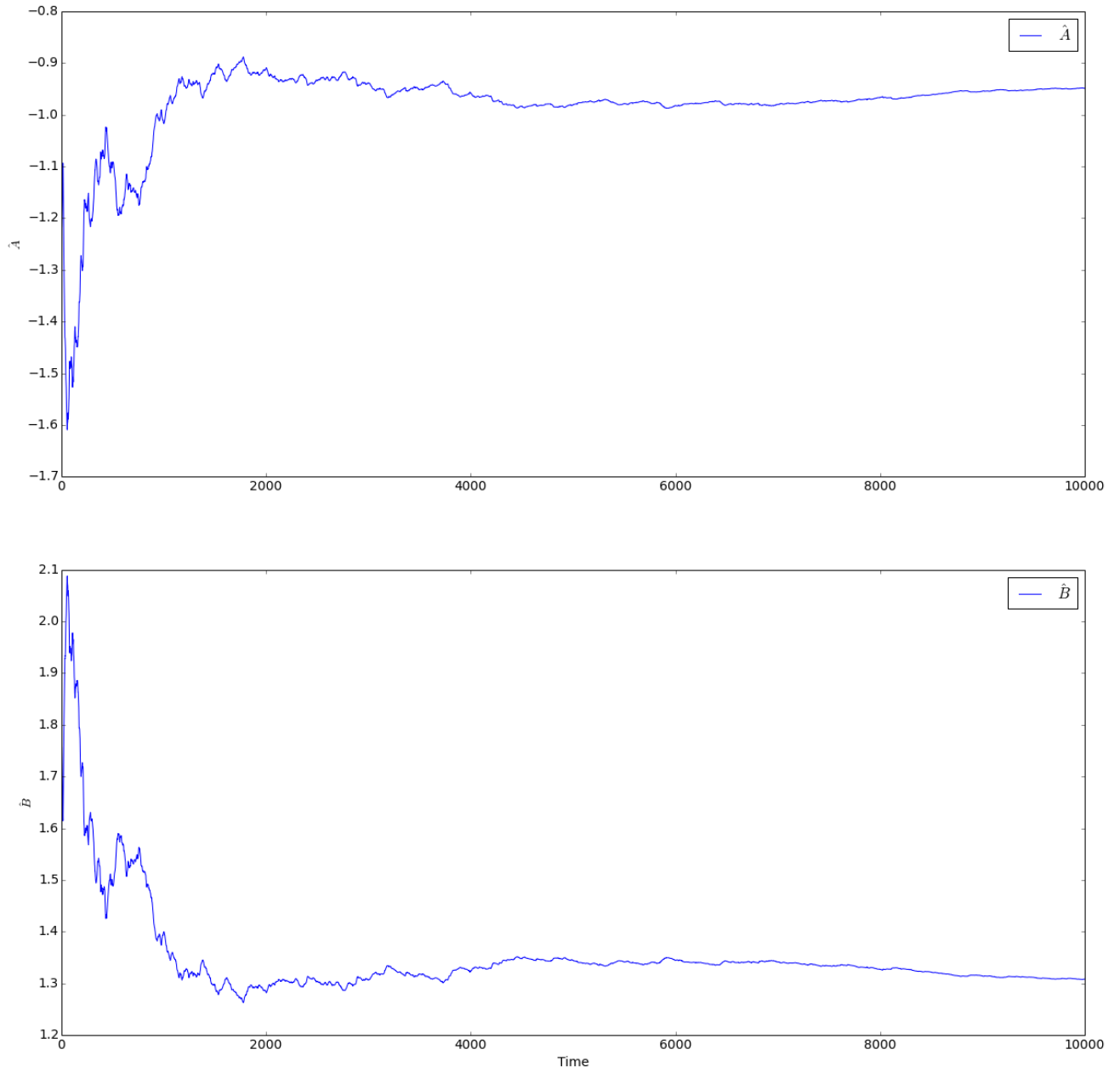
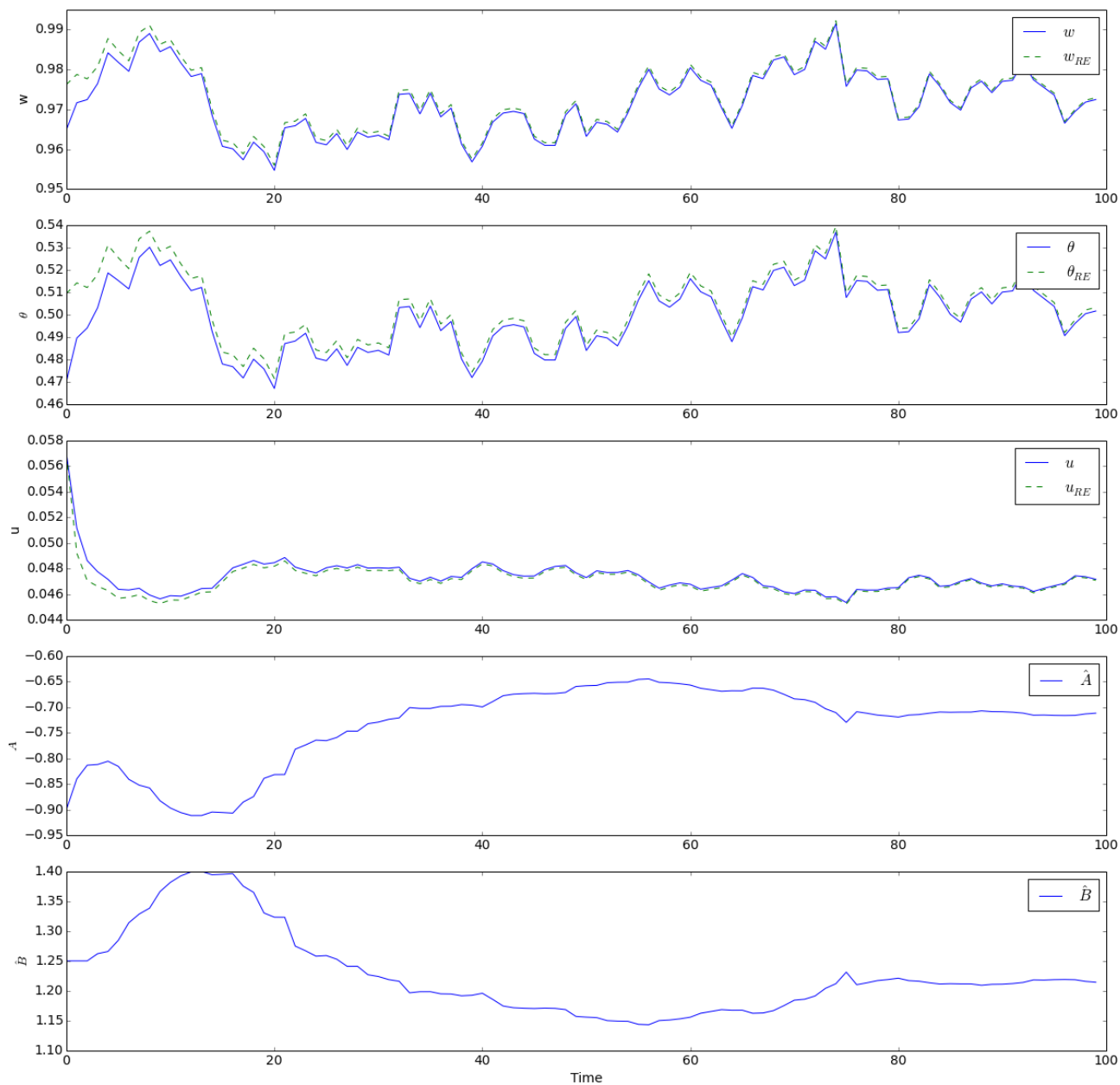
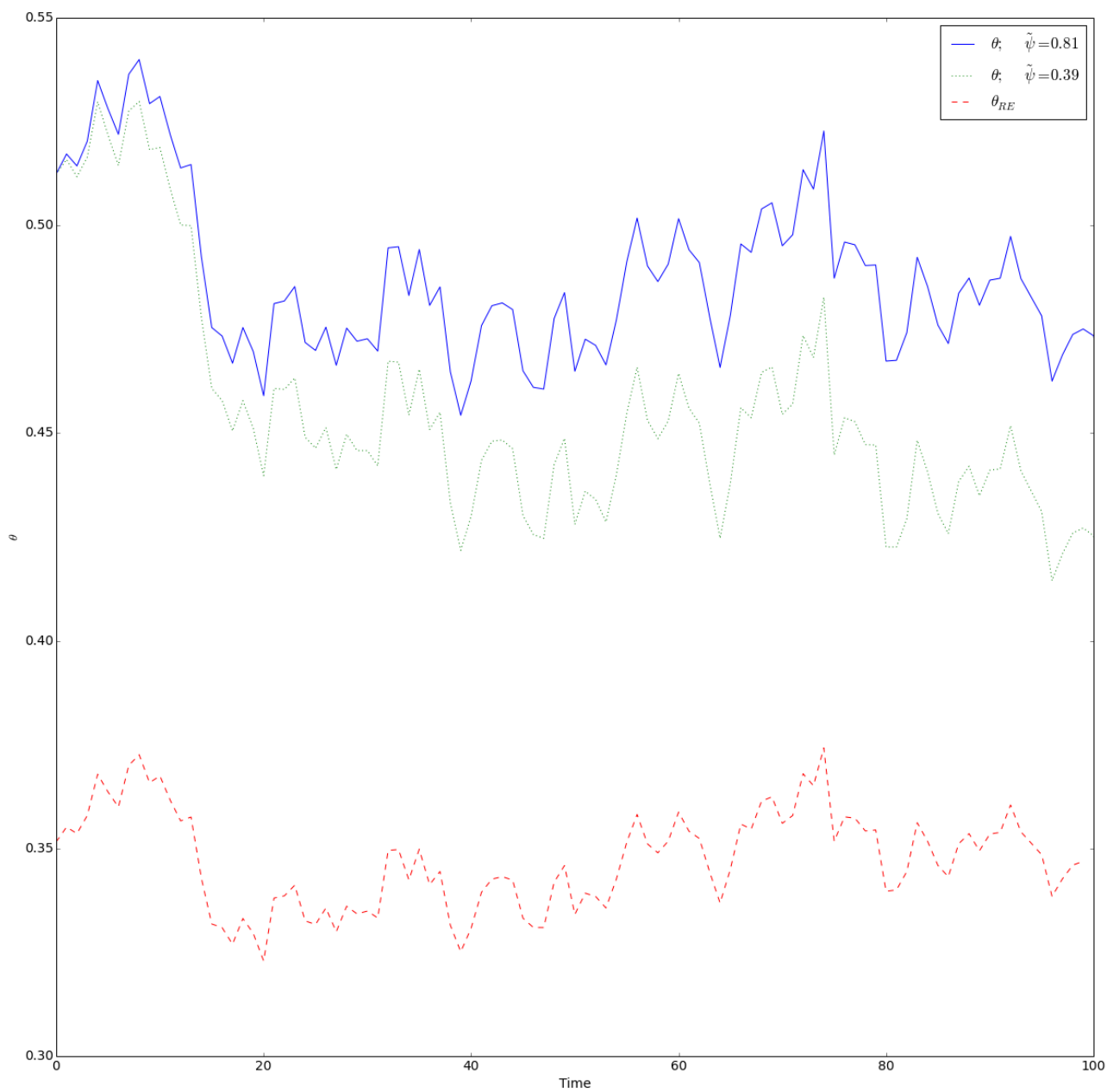


Figure 5: Learning (II) - Comparison of recursive least squares learning with rational expectations eq.



Note.- Following a -10% shock to the flow value of unemployment, with initial parameter estimates assumed to be at the true REE values.

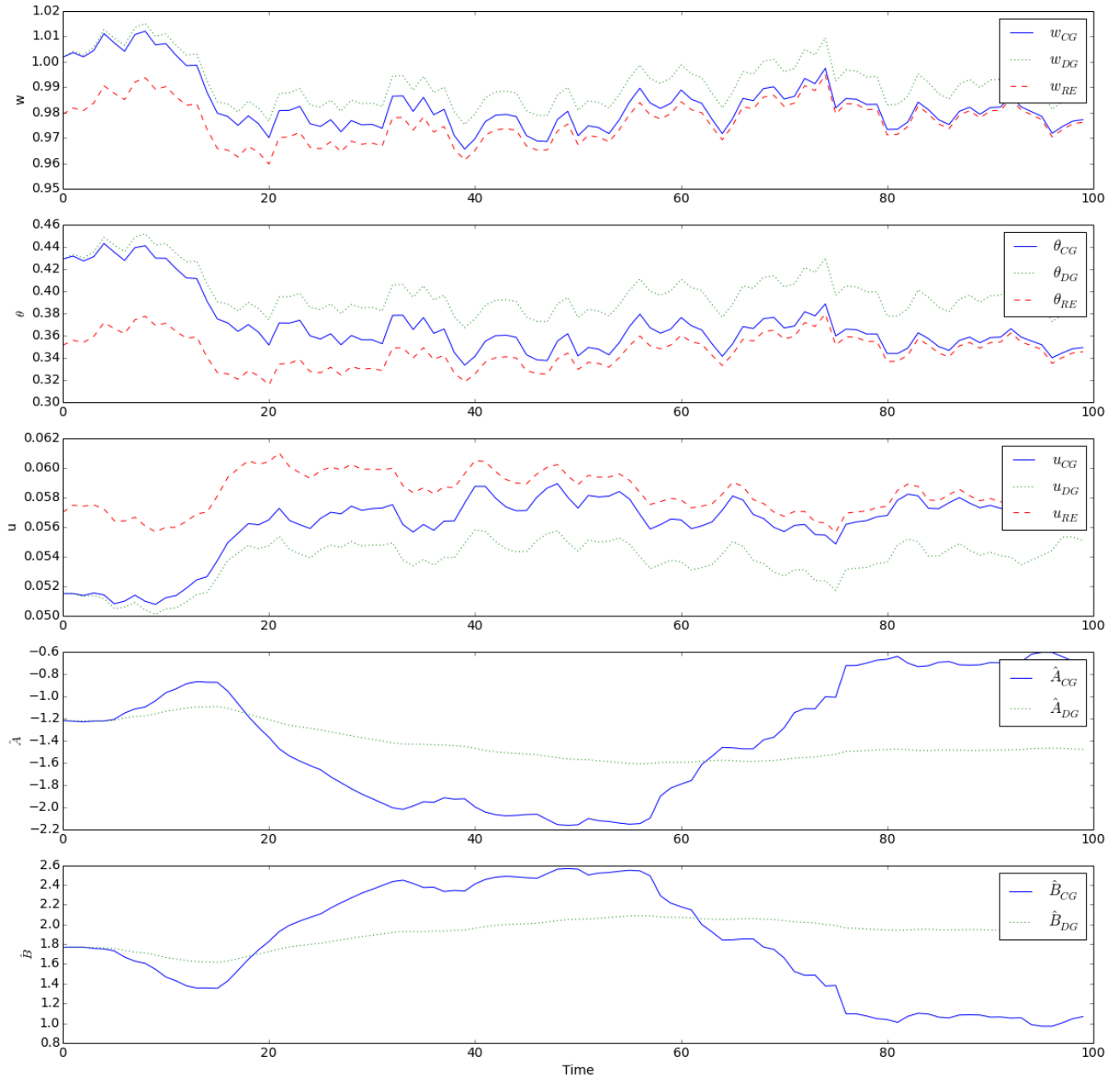
Figure 6: Learning (III) - Speed of convergence comparison, changing worker bargaining power  $\beta$



Note.- given  $\beta = 0.1$ , then  $\tilde{\psi} = 0.81$ . For  $\beta = \alpha$ ,  $\tilde{\psi} = 0.39$ .



Figure 7: Learning (IV) - Comparison of decreasing and constant gain learning



Note.- with initial parameter estimates assumed to be 'realistically' far away from true REE values.