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An axiomatization of difference-form contest success functions*

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May 3, 2015

Abstract

This paper presents an axiomatic characterization of difference-form group contests, that is, contests fought among groups and where their probability of victory depends on the difference of their effective efforts. This axiomatization rests on the property of Equalizing Consistency, stating that the difference between winning probabilities in the grand contest and in the smaller contest should be identical across all participants in the smaller contest. This property overcomes some of the drawbacks of the widely-used ratio-form contest success functions. Our characterization shows that the criticisms commonly-held against difference-form contests success functions, such as lack of scale invariance and zero elasticity of augmentation, are unfounded. By clarifying the properties of this family of contest success functions, this axiomatization can help researchers to find the functional form better suited to their application of interest.

Keywords: Contests, Groups, Contest success function, Axioms.
JEL codes: D31, D63, D72, D74.

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1 Introduction

Despite the relevance and ubiquity of contests in the real world, contest theory is often criticized for its great reliance on a particular construct: The Contest Success Function (Hirshleifer, 1989). This function maps the efforts made by contenders into their probability of attaining victory or, alternatively, their share of the contested prize. Critics argue that the CSF is too reduced form, too much of a black-box. For instance, the widely-used Tullock CSF (Tullock, 1967; 1980) under which success in the contest depends on relative efforts might seem sensible. But there is no apparent reason why this functional form should govern most types of contests, from interstate wars to sport competitions.[1] Because of this, the predictions of contest theory might be seen as too reliant on very specific functional forms rather than on sound economic principles.

This view is somewhat unfair for two reasons: Firstly, because there are other areas of Economics where very specific functional forms are often assumed. Secondly, because there is an active and fruitful strand of the literature which in the last few years has provided a variety of foundations to the most frequently employed CSFs.[2] This literature has even addressed the econometric estimation of these functions.[3] As a result of these efforts, economists have now at their disposal a growing menu of well-founded CSFs to choose from. The next natural question is which type of CSF is better suited to each specific application. A systematic study of the properties of each family of CSFs can contribute to that aim.

One family of contests assumes that winning probabilities depend on the difference of contenders' efforts. These *difference-form* contests were introduced by Hirshleifer (1989; 1991) and explored later by Baik (1998) and Che and Gale (2000) for the case of bilateral contests. Difference-form CSFs have been shown to emerge naturally in a number of settings. Gersbach and Haller (2009) show that a linear difference-form CSF is the result of intra-household bargaining when partners must decide how much time to devote to themselves or to their partner. Corchón and Dahm (2010) microfound a difference-form CSF as the result of a game where contenders are uncertain about the type of the external decider; by interpreting the CSF as a share, they also show that the difference-form coincides with the claim-egalitarian bargaining solution. Corchón and Dahm (2011) obtain the difference-form

[1] For excellent surveys of the contest literature see Corchon (2007) and Konrad (2009).

[2] These characterizations fall into four main categories: Axiomatic, stochastic, optimally-designed and microfounded (Jia, Skaperdas and Vaidya, 2013).

[3] For a detailed discussion of the econometric issues involved in the estimation of CSFs see Jia and Skaperdas (2011) and Jia et al. (2013).

as the result of a problem where the contest designer is unable to commit to a specific CSF once contenders have already exerted their efforts. Skaperdas and Vaydia (2012) show that the difference-form CSF can be derived in a Bayesian framework in which contenders produce evidence stochastically in order to persuade an audience of the correctness of their respective views. Finally, Polishchuk and Tonis (2013) show that a logarithmic difference-form CSF results from using a mechanism design approach when contestants have private information over their valuation of victory. In summary, it is fair to conclude that difference-form CSFs are by now well micro-founded. However, little is known about their actual properties and about how these differ from the properties of the more often used ratio-form CSFs, where winning probabilities are a function of the ratio of contenders' effective efforts.

The present paper offers the first axiomatic characterization of the family of difference-form CSFs. This axiomatization rests on the Equalizing Consistency property which imposes the following condition. Consider a smaller contest among a subset of participants in the grand contest. Equalizing Consistency imposes that the difference between winning probabilities in the grand contest and in the smaller contest should be identical across all participants in the smaller contest. Under this axiom then, disparities in winning probabilities across participants in a subcontest are smaller than in the grand contest. The Equalizing Consistency axiom differs from the consistency axiom employed in all the existing characterizations of the ratio-form CSF. That axiom, which we relabel as Proportional Consistency here, is typically understood as imposing that contestants' winning probabilities in a smaller contest should be proportional to their winning probability in the big contest. We employ here an equivalent interpretation of the axiom, which is that the difference between winning probabilities in the grand contest and in a smaller contest must be proportional to contestants' chances of success in the grand contest. Therefore, under Proportional Consistency, disparities in winning probabilities across participants in a smaller contest widen up compared to their winning probabilities in the grand contest.

We show that the Equalizing Consistency axiom can overcome some of the problems presented by the family of ratio-form CSFs. Our Theorem 1 shows that Equalizing Consistency, together with a number of reasonable axioms already employed in the literature, characterize a generalized version of the linear difference-form CSF introduced by Che and Gale (2000). This family of CSFs also encompass as particular cases the ones micro-founded in the aforementioned literature as well as the ones employed by Levine and Smith (1995), Rohner (2006), Besley and Persson (2008, 2009) and Gartzke and Rohner (2011).

With our axiomatization, we help to clarify the properties that characterize the families of CSFs studied in the literature. Contrary to the received wisdom, we show that the difference-form CSF can be scale invariant, i.e. homogeneous of degree zero (Theorem 2), and that it can have a positive elasticity of augmentation (Theorem 3).[4] These misconceptions are due to the common assumption of linear impacts, which we dispense with, and to the common usage of the term "difference-form CSF" to refer to the logistic functional form introduced by Hirshleifer (1989; 1991), under which winning probabilities are proportional to contenders' exponential efforts.

This paper contributes to the axiomatic work pioneered by Skaperdas (1996) and Clark and Riis (1998). Later, Münster (2009) extended this characterization from individual to group contests. Arbatskaya and Mialon (2009) and Rai and Sarin (2009) axiomatized multi-investment contests, whilst Blavatskyy (2010) did the same for contests with ties. More recently, Hwang (2012) axiomatized the family of CSF with constant elasticity of augmentation, which encompasses the logistic and the ratio forms as particular cases. Vesperoni (2013) and Lu and Wang (2014) axiomatized contests producing a ranking of players instead of a sole winner. Lu and Wang (2014) characterized success functions for contests with strict rankings of players, whereas Vesperoni (2013) axiomatized an alternative success function for rankings of any type. Finally, Bozbay and Vesperoni (2014) characterized a CSF for conflicts embedded in network architectures. Let us add that in our axiomatization we make connections with the income inequality literature, and in particular with the concept of absolute inequality introduced by Kolm (1976a,b). The literature on inequality measurement offers valuable insights on the properties of functional forms which we explicitly employ at several points of the text.[5]

2 Axiomatization

In order to be as general as possible, we consider a society divided in $K \geq 2$ disjoint groups formed by a number $n_k \geq 1$ of individuals each adding up to a total of N.[6] Denote the set of groups by $\mathbb{K}$. These K groups are in

[4] Elasticity of augmentation was introduced by Hwang (2012). A positive elasticity implies that the difference between the winning probabilities of two contenders diminishes when their efforts increase whilst keeping constant the difference between them.

[5] In this same spirit, Chakravarty and Maharaj (2014) have recently characterized a new family of individual contests success functions which satisfy properties akin to the intermediate inequality and ordinal consistency axioms employed in the income distribution literature.

[6] Individual contests are a particular case of the ones studied here. All our results, except those in Section 4, which deal with the aggregation of individual efforts within

competition. They are engaged in a contest which can have only one winner. Members of the contender groups can expend non-negative effort in order to help their group to win the contest. Depending on the specific type of contest, these efforts can be money, time, physical effort or weapons. Denote by $\mathbf{x}_k \equiv (x_{1k}, ..., x_{n_k k}) \in \mathbb{R}_+^{n_k}$ the vector of non-negative efforts by members of group k and by $\mathbf{x}$ the vector $(\mathbf{x}_1, ..., \mathbf{x}_K)$. For convenience we will denote by $\mathbf{x}_{-k}$ the vector of efforts in groups other than k.

Efforts determine the winning probability of each group according to a Contest Success Function (CSF) $p_k : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$. The function $p_k(\mathbf{x})$ can also be thought of as the share of the prize or object being contested that group k obtains in case of victory. We favor the former interpretation throughout the paper.

Let us now state the axioms that we would like to impose on our CSF.

2.1 Two basic axioms: Let us first present two axioms introduced by Skaperdas (1996) in his axiomatization of CSFs for individual contests and later generalized by Münster (2009) for group contests. These axioms are rather natural and should thus apply to the class of difference-form group contests we study in this paper.

Axiom 1 (Probability) $\sum_{k=1}^{K} p_k(\mathbf{x}) = 1$ *and* $p_k(\mathbf{x}) \geq 0$ *for any* $\mathbf{x}$ *and all* $k \in \mathbb{K}$.

Axiom 2 (Monotonicity) *Consider two generic vectors* $\mathbf{x}_k$ *and* $\mathbf{x}'_k$ *such that* $\mathbf{x}'_k > \mathbf{x}_k$. *Then,*

(i) $p_k(\mathbf{x}'_k, \mathbf{x}_{-k}) \geq p_k(\mathbf{x}_k, \mathbf{x}_{-k})$, *with strict inequality if* $p_k(\mathbf{x}_k, \mathbf{x}_{-k}) \in (0, 1)$.

(ii) $p_l(\mathbf{x}'_k, \mathbf{x}_{-k}) \leq p_l(\mathbf{x}_k, \mathbf{x}_{-k})$ *for all* $l \neq k$ *and* $l \in \mathbb{K}$.

The axiom of Probability just states that the CSF generates a probability distribution over the set of groups. The Monotonicity axiom implies that group winning probabilities are weakly increasing in the effort of members and weakly decreasing in the effort of outsiders. Note that this axiom is slightly weaker than the Monotonicity axiom employed in Münster (2009) and the analogous one in Rai and Sarin (2009).

2.2 Subcontest axioms: The next two axioms relate to contests played among a generic non-empty subset $\mathbb{S} \subset \mathbb{K}$ of contender groups. We refer to this contest among groups in $\mathbb{S}$ as a *subcontest*. Let us denote by S the

groups, thus apply to individual contests as well.

cardinality of subcontest $\mathbb{S}$ and by $p_k^{\mathbb{S}}(\mathbf{x})$ the winning probability of group k in this subcontest. In particular, denote by $p_k^{\{k,l\}}(\mathbf{x})$ the winning probability of group k in the bilateral contest against group l. Finally, denote by $\mathbf{x}_{\mathbb{S}}$ and $\mathbf{x}_{-\mathbb{S}}$ the vector of efforts in the groups inside and outside $\mathbb{S}$ respectively.

Axiom 3 (Independence) *$p_k^{\mathbb{S}}(\mathbf{x})$ does not depend on $\mathbf{x}_{-\mathbb{S}}$; or $p_k^{\mathbb{S}}(\mathbf{x})$ can be written as $p_k^{\mathbb{S}}(\mathbf{x}_{\mathbb{S}})$.*

Independence implies that the efforts made by contenders outside a subcontest should not matter to its result. As discussed by Skaperdas (1996) and Clark and Riis (1998), this property relates to the axiom of Independence of Irrelevant Alternatives in probabilistic individual choice. Thus, it is a reasonable property in contests where nature determines the winner. Independence also implies that there are no spillovers across subcontests or that spillovers affect all contenders in $\mathbb{S}$ equally.[7]

The next axiom is crucial in our axiomatization of the family of difference-form CSFs.

Axiom 4 (Equalizing Consistency) *For any vector $\mathbf{x}$ and any subcontest $\mathbb{S} \subset \mathbb{K}$ such that $p_k(\mathbf{x}) > 0$ for all $k \in \mathbb{S}$*

$$p_k^{\mathbb{S}}(\mathbf{x}) = p_k(\mathbf{x}) + \frac{1}{S}(1 - \sum_{l \in \mathbb{S}} p_l(\mathbf{x})). \quad (1)$$

Given any subcontest formed by groups with a positive probability of winning the grand contest, Equalizing Consistency states that the probability of group k winning that subcontest is equal to its probability of winning the grand contest plus a fixed amount. That amount is an equal proportion of the probability that no group in the subcontest wins the grand contest. In other words, suppose that a number of contenders drop out from $\mathbb{K}$ and that all remaining contenders had a positive probability of winning the grand contest. The axiom states that the remaining groups "share" equally the probability of any of the non participating groups winning the grand contest.

It is natural to expect that the winning probability of contenders should be higher when competing in $\mathbb{S}$ than when competing in $\mathbb{K}$. Equalizing Consistency implies that this increase should be the same across contenders. This, of course, does not exhaust all possibilities. Interestingly, it is possible to state the main axiom characterizing the family of ratio-form CSFs in a similar way. This axiom is called simply Consistency by Skaperdas

[7] We thank Luis Corchon for pointing this out.

(1996), Münster (2009) and Ray and Sarin (2009). We here rename it as Proportional Consistency in order to avoid confusion with the previous axiom.

Axiom 4' (Proportional Consistency) *For any vector* $\mathbf{x}$ *and any group* $k \in \mathbb{S} \subset \mathbb{K}$

$$p_k^{\mathbb{S}}(\mathbf{x}) = \frac{p_k(\mathbf{x})}{\sum_{l \in \mathbb{S}} p_l(\mathbf{x})} = p_k(\mathbf{x}) + \frac{p_k(\mathbf{x})}{\sum_{l \in \mathbb{S}} p_l(\mathbf{x})}(1 - \sum_{l \in \mathbb{S}} p_l(\mathbf{x})). \tag{2}$$

This axiom posits that the increase in winning probabilities that members of the subcontest experience from narrowing the contest from $\mathbb{K}$ to $\mathbb{S}$ must be proportional to their winning probability in the grand contest.

Let us now devote some time to compare the implications of these two axioms. Proportional Consistency is often invoked as a natural assumption. It implies that the relative success of two groups should be identical across subcontests, that is

$$\frac{p_k^{\mathbb{S}}(\mathbf{x})}{p_l^{\mathbb{S}}(\mathbf{x})} = \frac{p_k(\mathbf{x})}{p_l(\mathbf{x})}.$$

However, this property presents some drawbacks. First, it is not well defined when $p_k(\mathbf{x}) = 0$ for all contenders in $\mathbb{S}$. Second, it forces contenders with zero probability in the grand contest to have a zero winning probability in any subconstest. For instance, suppose that a contender k is very weak and has a zero winning probability in the big contest, whereas group l is marginally stronger and has a winning probability ε arbitrarily close to zero. Then group k must have a zero winning probability in any subcontest $\mathbb{S}$, including the bilateral contest against the similarly weak group l. This might be undesirable in some contexts, as in Political Economy applications, where a party may have no chances in a general election but a large probability of winning a local one.

On the other hand, the Equalizing Consistency axiom implies that the difference in winning probabilities between two groups should be identical across subcontests whose groups have positive winning probabilities in the grand contest, that is

$$p_k^{\mathbb{S}}(\mathbf{x}) - p_l^{\mathbb{S}}(\mathbf{x}) = p_k(\mathbf{x}) - p_l(\mathbf{x}).$$

Therefore, Equalizing Consistency does not impose anything on contenders with a zero winning probability in $\mathbb{K}$. Still, we can compare both axioms when a contender has a winning probability arbitrarily close to zero. Take two contenders, one with winning probability $p < 1$ and a weaker one with winning probability $\varepsilon < p$. Under Proportional Consistency, the latter would win their bilateral contest with probability $\frac{\varepsilon}{\varepsilon+p}$, which tends to zero as ε goes to zero. Under Equalizing Consistency though, the weaker contender would win instead with probability $\frac{1}{2} - \frac{p-\varepsilon}{2}$ which tends to $\frac{1-p}{2} > 0$ as ε goes to zero.

The reader may argue that one drawback of the Equalizing Consistency axiom could be the following: Consider a scenario where one contender group is much weaker than the rest of groups, who are all equally strong. But because there is a large number of these strong groups, each of them enjoys a winning probability of just $q\varepsilon$ where $q > 1$ and ε is the winning probability of the weak group. Equalizing Consistency would imply that the weak group would have a winning probability of $\frac{1}{2} - \frac{(q-1)\varepsilon}{2}$ in a bilateral contest against one of the strong groups. This seems unrealistic since this group is substantially weaker than the other. But as we will see below, the CSF that we axiomatize here would bound to zero the winning probability of the weak contender; if this group had a positive winning probability in the grand contest then it could not be much weaker than the rest.

One last word on the comparison of these two axioms. Equalizing and Proportional Consistency are approximately equivalent when the members of the subcontest have similar winning probabilities in the grand contest, so that $\frac{p_k(\mathbf{x})}{\sum_{l \in \mathbb{S}} p_l(\mathbf{x})} \approx \frac{1}{S}$.

2.3 The main theorem: We are now in the position to state our main theorem characterizing the family of the difference-form CSFs. This family emerges from using the basic axioms of Probability, Monotonicity and Independence together with Equalizing Consistency.

Theorem 1 *If the CSF $p_k(\mathbf{x})$ is continuous and satisfies axioms A1-A4 then for each vector $\mathbf{x}$ there exists an integer $K^* \leq K$ such that*

$$p_k(\mathbf{x}) = \begin{cases} \frac{1}{K^*} + h_k(\mathbf{x}_k) - \frac{1}{K^*} \sum_{l=1}^{K^*} h_l(\mathbf{x}_l) & \text{for } k = 1, ..., K^* \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where without loss of generality $h_{k+1}(\mathbf{x}_{k+1}) \leq h_k(\mathbf{x}_k)$ and where each $h_k : \mathbb{R}_+^{n_k} \to \mathbb{R}$ is a continuous and increasing function.

Proof. Take any three contender groups j, k and l with strictly positive winning probabilities in $\mathbb{K}$. Then,

$$\begin{aligned} p_k(\mathbf{x}) - p_l(\mathbf{x}) &= p_k(\mathbf{x}) - p_j(\mathbf{x}) + p_j(\mathbf{x}) - p_l(\mathbf{x}) \\ &= p_k^{\{k,j\}}(\mathbf{x}_k, \mathbf{x}_j) - p_j^{\{k,j\}}(\mathbf{x}_k, \mathbf{x}_j) + p_j^{\{l,j\}}(\mathbf{x}_l, \mathbf{x}_j) - p_l^{\{l,j\}}(\mathbf{x}_l, \mathbf{x}_j), \end{aligned}$$

where the second equality comes from A3 and from the fact that A4 also applies to the bilateral contests involving the pairs of contenders $\{k,j\}$ and $\{l,j\}$. By the same token, it must be that $p_k(\mathbf{x}) - p_l(\mathbf{x}) = p_k^{\{k,l\}}(\mathbf{x}_k, \mathbf{x}_l) - p_l^{\{k,l\}}(\mathbf{x}_k, \mathbf{x}_l)$ so it is possible to rewrite the above expression as

$$\begin{aligned} p_k^{\{k,l\}}(x_k, x_l) - p_l^{\{k,l\}}(x_k, x_l) &= p_k^{\{k,j\}}(x_k, x_j) - p_j^{\{k,j\}}(x_k, x_j) \\ &\quad -(p_l^{\{l,j\}}(x_l, x_j) - p_j^{\{l,j\}}(x_l, x_j)), \end{aligned}$$

which by A1 boils down to

$$p_k^{\{k,l\}}(\mathbf{x}_k, \mathbf{x}_l) - p_l^{\{k,l\}}(\mathbf{x}_k, \mathbf{x}_l) = 2(p_k^{\{k,j\}}(\mathbf{x}_k, \mathbf{x}_j) - p_l^{\{l,j\}}(\mathbf{x}_l, \mathbf{x}_j)).$$

Since the left hand side of the expression does not depend on $\mathbf{x}_j$, the right hand side cannot depend on $\mathbf{x}_j$ either. Therefore, we can rewrite the right hand side as the difference of two functions

$$p_k^{\{k,l\}}(\mathbf{x}_k, \mathbf{x}_l) - p_l^{\{k,l\}}(\mathbf{x}_k, \mathbf{x}_l) = h_k(\mathbf{x}_k) - h_l(\mathbf{x}_l).$$

And by using A4 again we obtain

$$p_k(\mathbf{x}) - p_l(\mathbf{x}) = h_k(\mathbf{x}_k) - h_l(\mathbf{x}_l). \tag{4}$$

Note that $h_k : \mathbb{R}_+^{n_k} \to \mathbb{R}$ must be continuous given that $p_k(\mathbf{x})$ and $p_l(\mathbf{x})$ are continuous too. Denote by $\mathbb{K}^*$ the set of contender groups who enjoy a positive winning probability in the grand contest, that is, those A4 apply to. Let K^* be the cardinality of this set. Adding up expression (4) across these K^* groups we obtain

$$K^* p_k(\mathbf{x}) - 1 = K^* h_k(\mathbf{x}_k) - \sum_{l \in \mathbb{K}^*} h_l(\mathbf{x}_l), \tag{5}$$

which after noting that $\sum_{l \in \mathbb{K}^*} p_l(\mathbf{x}) = 1$ leads to expression (3).

We next characterize the set $\mathbb{K}^*$. We have established that p_k can be written as a mapping from impacts into probabilities, i.e. $p_k : \mathbb{R}_+^K \to [0,1]$. We can then associate a function $h_k : \mathbb{R}_+^{n_k} \to \mathbb{R}$ to each group. Given $\mathbf{x}$ and without loss of generality, let us order groups in a decreasing manner such that $h_k(\mathbf{x}_k) \geq h_{k+1}(\mathbf{x}_{k+1})$. We shall now argue that the set $\mathbb{K}^*$ is given by the largest integer K^* such that

$$\frac{1}{K^*} + h_{K^*}(\mathbf{x}_{K^*}) - \frac{1}{K^*} \sum_{l=1}^{K^*} h_l(\mathbf{x}_l) > 0. \tag{6}$$

Because we ordered groups in a decreasing manner, (6) holds true for any group $k = 1, ..., K^*$. Observe that such K^* is unique and well-defined since it is straightforward to show that

$$\frac{1}{K^*}+h_{K^*}(\mathbf{x}_{K^*})-\frac{1}{K^*}\sum_{l=1}^{K^*} h_l(\mathbf{x}_l) > \frac{1}{K^*+1}+h_{K^*+1}(\mathbf{x}_{K^*+1})-\frac{1}{K^*+1}\sum_{l=1}^{K^*+1} h_l(\mathbf{x}_l).$$

By construction, (6) must hold for any contender group with $p_k(\mathbf{x})$ >0. We must then work out the opposite implication, that is, that if (6) holds for group k then $p_k(\mathbf{x})$ >0. Suppose on the contrary that $p_k(\mathbf{x})$ =0. Adding up (3) across all groups l from 1 to K^* except k yields

$$\begin{aligned} 1 &= \frac{K^*-1}{K^*} + \sum_{l=1,l\neq k}^{K^*} h_l(\mathbf{x}_l) - \frac{K^*-1}{K^*}\sum_{l=1}^{K^*} h_l(\mathbf{x}_l) \\ &= \frac{K^*-1}{K^*} - h_k(\mathbf{x}_k) + \frac{1}{K^*}\sum_{l=1}^{K^*} h_l(\mathbf{x}_l), \end{aligned}$$

implying that

$$\frac{1}{K^*} + h_k(\mathbf{x}k) - \frac{1}{K^*}\sum_{l=1}^{K^*} h_l(\mathbf{x}_l) = 0,$$

thus contradicting that (6) holds for k.

We must finally prove that each function $h_k(\mathbf{x}_k)$ is increasing. Consider a pair of vectors $\mathbf{x}'$ and $\mathbf{x}$ such that $\mathbf{x}' = (\mathbf{x}_1, ..., \mathbf{x}'_k, ..., \mathbf{x}_K)$ where $\mathbf{x}'_k > \mathbf{x}_k$. That is, vector $\mathbf{x}'$ is identical to vector $\mathbf{x}$ except for group k. By A2 it must be that $p_k(\mathbf{x}) \leq p_k(\mathbf{x}')$ and $p_l(\mathbf{x}) \geq p_l(\mathbf{x}')$ for any $l \neq k$. The property holds trivially if $p_k(\mathbf{x}) = 0$ or $p_l(\mathbf{x}) = 0$. If both probabilities are strictly positive, expression (4) implies that

$$\begin{aligned} h_k(\mathbf{x}_k)-h_l(\mathbf{x}_l) &= p_k(\mathbf{x})-p_l(\mathbf{x}) \\ &\leq p_k(\mathbf{x}') - p_l(\mathbf{x}') = h_k(\mathbf{x}'_k)-h_l(\mathbf{x}_l), \end{aligned}$$

thus proving that $h_k(\mathbf{x}_k)$ is increasing. This finalizes the proof. ■

The function $h_k(\mathbf{x}_k)$ is commonly known as the *impact function.* It aggregates members' efforts into a measure of their group influence in the contest. Alternatively, it can be seen as the function determining how effective players' efforts are.

The difference-form group CSF in (3) relates the success of a group to the difference between its impact and the average impact of all the groups involved in the contest. If the impact of a group is above (below) the average impact, its winning probability must be above (below) the winning probability that the group would be awarded under a fair lottery. If the impact of a group is positive but sufficiently below average impact, its winning probability is just zero. Of course, this should never happen in equilibrium if efforts are costly.

One immediately obvious feature of this CSF is that it is additively separable in the impact of the contestant groups. The marginal productivity of individual efforts does not depend on the efforts of outsiders. This implies that any equilibrium in an individual contest under this CSF must be in dominant strategies.

The difference-form CSF also allows contenders to attain a sure victory if their impact is sufficiently large. Take for instance, the case of two-group contests, i.e. $K = 2$, where (3) boils down to

$$p_k(\mathbf{x}) = \min\left\{\max\left\{\frac{1}{2} + \frac{\beta}{2}[h_k(\mathbf{x}_k) - h_j(\mathbf{x}_j)], 0\right\}, 1\right\}.$$

This CSF generalizes the linear difference-form CSF for two-player contests introduced by Che and Gale (2000) and later employed by Rohner (2006), Besley and Persson (2008, 2009) and Gartzke and Rohner (2011). Contender k can obtain victory with certainty if $h_k(\mathbf{x}_k) \geq \frac{1}{\beta} + h_j(\mathbf{x}_j)$. This shows that, as highlighted by Che and Gale (2000), the difference-form CSF has strong connections with auctions. Albeit noisy, this CSF allows contenders to obtain a sure win by outbidding others by a wide enough margin. On the other hand, a contender with zero impact can still enjoy a positive winning probability if the other contestants have not too large impacts. That would be the case for k in the example above if $h_j(\mathbf{x}_j) < \frac{1}{\beta}$. As argued by Hirshleifer (1991) this fits well with confrontations with severe frictions such as incomplete information, fatigue or difficult terrain where contenders find extremely costly to overpower rivals.

An final remark is in order here: Our Theorem 1 differs from Theorem 1 in Münster (2009) in that we replace Proportional Consistency by Equalizing Consistency. This new axiom leads to the characterization of the family of difference-form CSFs. Recall that Proportional and Equalizing Consistency differ on how contenders' winning probability in a subcontest relate to their winning probability in the grand contest. In particular, they differ on how to distribute -proportionally or equally- the "excess" probability $1 - \sum_{l \in \mathbb{S}} p_l(\mathbf{x})$ across members of the subcontest. This can explain why Corchon and Dahm (2010) find that the difference-form CSF generates a sharing rule which coincides with the claim-egalitarian bargaining solution whereas the ratio-form CSF coincides with the claim-proportional bargaining solution.

3 Invariance

3.1 Scale invariance

In this section, we study two other properties employed in previous axiomatic characterizations of CSFs. These properties impose the invariance of winning probabilities to certain changes in the profile of contestants' efforts. The first one, and most-commonly used, is homogeneity of degree zero, which we refer to here as scale invariance.

Axiom 5 (Scale Invariance) *For all $\lambda > 0$ and all $k \in \mathbb{K}$*

$$p_k(\lambda \mathbf{x}) = p_k(\mathbf{x}).$$

This axiom states that winning probabilities must remain constant to equiproportional changes in all contenders' efforts. Scale invariance implies that units of measurement of effort do not matter. This is a desirable property when efforts are measured in money or military units (battalions, regiments, etc.). It is a property which is also satisfied by the indices of relative inequality introduced by Atkinson (1970).

Münster (2009) proved that if a CSF satisfies axioms A1-A3, A4' and A6, then the impact functions in the ratio-form CSF must be all homogeneous of the same degree. Let us now do the analogous axiomatization exercise in our setting and characterize the family of scale invariant difference-form CSFs.

Theorem 2 *If a CSF satisfies axioms A1-A5, then it satisfies (3) and the impact functions $h_k(\mathbf{x}_k)$ are homothetic functions satisfying*

$$h_k(\mathbf{x}_k) = \alpha_k + \beta \ln g(\mathbf{x}_k), \tag{7}$$

where α_k and *$\beta > 0$ are parameters and the function $g(\mathbf{x}_k) : \mathbb{R}_+^{n_k} \rightarrow \mathbb{R}_+$ is increasing, homogeneous of degree one and satisfies $g(\mathbf{0}) = 0$* and *$g(\mathbf{1}) = 1$.*

Proof. A5 implies that $p_k(\lambda \mathbf{x}) = p_k(\mathbf{x})$. Hence, if $p_k(\mathbf{x}) = 0$ then $p_k(\lambda \mathbf{x}) = 0$ and viceversa, so the set $\mathbb{K}^*$ does not change. Again, we denote its cardinality simply by K^*.

In the next step of the proof we follow a similar procedure as in the proof of Theorem 2 in Rai and Sarin (2009, p. 147). Take any two different groups

$k, j \leq K^*$. By Theorem 1, $p_k(\lambda \mathbf{x}) = p_k(\mathbf{x})$ implies that their impact functions satisfy

$$h_k(\lambda \mathbf{x}_k) - h_k(\mathbf{x}_k) = h_j(\lambda \mathbf{x}_j) - h_j(\mathbf{x}_j) = \frac{1}{K^*}[\sum_{l \in \mathbb{K}} h_l(\lambda \mathbf{x}_l) - \sum_{l \in \mathbb{K}} h_l(\mathbf{x}_l)],$$

for any $\mathbf{x}_k \in \mathbb{R}_+^{n_k} \backslash \{\mathbf{0}\}$. Since the last term in the above equality is just a constant, the difference $h_k(\lambda \mathbf{x}_k) - h_k(\mathbf{x}_k)$ is the same for all $k \leq K^*$ and we can conclude that this difference depends on λ but not on $\mathbf{x}_k$. Hence it must hold true that

$$h_k(\lambda \mathbf{x}_k) - h_k(\mathbf{x}_k) = h_k(\lambda \cdot \mathbf{1}) - h_k(\mathbf{1}),$$

where $\mathbf{1} = (1, ..., 1)$ is the vector of appropriate length whose components are all equal to one.

Now add and substract $h_k(\mathbf{1})$ to the left hand side of this expression and denote $H(\mathbf{x}_k) = h_k(\mathbf{x}_k) - h_k(\mathbf{1})$. It can then be rewritten as

$$H(\lambda \mathbf{x}_k) = H(\lambda \mathbf{1}) + H(\mathbf{x}_k).$$

If $\mathbf{x}_k = t \cdot \mathbf{1}$ for $t > 0$ then

$$H(\lambda t \mathbf{1}) = H(\lambda \mathbf{1}) + H(t \mathbf{1}).$$

Define now $G(\lambda) = H(\lambda \mathbf{1})$. This is a function of just one variable and it is increasing and continuous since by Theorem 1 we know already that $h_k(\mathbf{x}_k)$ must be increasing and continuous. We can then rewrite

$$G(\lambda t) = G(\lambda) + G(t).$$

This is one of the Cauchy functional equations whose only solution is given by $G(z) = \beta \ln z$ where β is an arbitrary constant (Aczél, 1966, p. 41). This implies

$$H(\lambda \mathbf{1}) = G(\lambda) = \ln \lambda^\beta,$$

and by the same token that

$$H(\lambda \mathbf{x}_k) = \ln \lambda^\beta + H(\mathbf{x}_k).$$

Given our definition of the function $H(\cdot)$, this implies that

$$h_k(\lambda \mathbf{x}_k) = \ln \lambda^\beta + h_k(\mathbf{x}_k).$$

By A5, it must be that β is identical for all impact functions.

Now, consider the case $\mathbf{x}_k = \mathbf{0}$. Fix $\overline{\mathbf{x}}_j \neq \mathbf{0}$ for each $j \neq k$ in $\mathbb{K}^*$ and $\lambda \neq 1$. Because $p_k(\lambda\mathbf{x})$ =$p_k(\mathbf{x})$ it must be that

$$\begin{aligned} & \frac{1}{K^*} + \frac{K^*-1}{K^*} h_k(\mathbf{0}) - \frac{1}{K^*} \sum_{l \leq K^*, l \neq k} (\ln \lambda^\beta + h_l(\mathbf{x}_l)) \\ = \ & \frac{1}{K^*} + \frac{K^*-1}{K^*} h_k(\mathbf{0}) - \frac{1}{K^*} \sum_{l \leq K^*, l \neq k} h_l(\mathbf{x}_l). \end{aligned}$$

This identity can only hold under two circumstances: First, if $\beta = 0$, which in turn implies that the impact functions must be homogeneous of degree zero, i.e. $h_k(\lambda\mathbf{x}_k) = h_k(\mathbf{x}_k)$. But for any $\lambda \neq 1$ this violates A2 since it imposes that the impact functions must be strictly increasing in their arguments when a winning probability is in the interval $(0, 1)$. So we are left with the only other possible case, that is, $\lim_{\mathbf{x}_k \to \mathbf{0}^+} h_k(\mathbf{x}_k) = -\infty$.

Next define $F_k(\mathbf{x}_k) = \exp\{h_k(\mathbf{x}_k)\}$. It is clear that the function $F_k(\mathbf{x}_k)$ is homogeneous of degree β since $F_k(\lambda\mathbf{x}_k) = \lambda^\beta F_k(\mathbf{x}_k)$. This is a function of one variable, which in turn must be a multiple of a power function, i.e. $F(s) = as^\beta$ with $\beta > 0$ and $a = F(1)$ (Münster, 2009; p 352). Hence it is possible to rewrite

$$F_k(\mathbf{x}_k) = a_k(g_k(\mathbf{x}_k))^\beta,$$

where $g_k : \mathbb{R}_+^{n_k} \longrightarrow \mathbb{R}$. The function $g_k(\mathbf{x}_k)$ must be homogeneous of degree one since

$$F_k(\lambda\mathbf{x}_k) = \lambda^\beta F_k(\mathbf{x}_k) \Rightarrow g_k(\lambda\mathbf{x}_k) = \lambda g(\mathbf{x}_k).$$

Finally, tracing back our steps

$$h_k(\mathbf{x}_k) = \ln F_k(\mathbf{x}_k) = \ln a_k + \beta \ln g(\mathbf{x}_k) = \alpha_k + \beta \ln g(\mathbf{x}_k).$$

Given that $\lim_{\mathbf{x}_k \to \mathbf{0}^+} h_k(\mathbf{x}_k) = -\infty$ it must be that $g(\mathbf{0}) = 0$. Finally, observe that it must also be that $g_k(\mathbf{1}) = 1$ given that

$$a_k = F_k(1) = \exp\{h_k(\mathbf{1})\} = a_k(g_k(\mathbf{1}))^\beta.$$

■

The difference-form CSF has been often criticized because it seemed to necessarily violate scale-invariance (Skaperdas, 1996; Hirshleifer, 2000; Alcalde and Dahm, 2007, p. 103; Corchón, 2007, p. 74). Our Theorem 2 proves that such criticism is ungrounded. If the impact function is of the form (7), winning probabilities under the family of difference-form CSFs in (3) are invariant to equiproportional changes in contenders' efforts. Changes in the unit of measurement of efforts do not generate changes in contenders' winning probabilities (albeit their impact do change).

To the best of our knowledge, this family of scale invariant difference-form CSFs has only been studied in Polishchuk and Tonis (2013, p. 218). They microfound a CSF of the form

$$p_k(\mathbf{x}) = \frac{1}{K} + \ln g(x_k) - \frac{1}{K}\sum_{l=1}^{K} \ln g(x_l),$$

by using a mechanism design approach when contenders are individuals who have private information over their valuation of victory. For the case of group contests, one example of a function satisfying (7) is the function $h_k(\mathbf{x}_k) = \alpha_k + \beta \ln(\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik})$, which we study in more detail in Section 4.

3.2 Translation invariance

If a CSF is defined as a function of the difference between contenders' efforts, another natural invariance property is the following: Winning probabilities should remain constant when the effort of all contenders increase by the same amount. This is equivalent to the following property.

Axiom 6 (Translation Invariance) *For all $\lambda > 0$ and all $k \in \mathbb{K}$,*

$$p_k(\mathbf{x}+\lambda\cdot\mathbf{1}) = p_k(\mathbf{x}).$$

Skaperdas (1996) and Münster (2009) used this property as an alternative to homogeneity of degree zero in their axiomatization of the ratio-form CSFs. Actually, Translation Invariance can be traced back to the income distribution literature, and in particular to the concept of absolute inequality introduced by Kolm (1976a,b). Absolute inequality states that the level of inequality in a distribution should not vary when the income of every individual increases by the same fixed amount. Hence, any measure of absolute income inequality must be translation invariant.

However, the standard Translation Invariance axiom builds in an implicit bias against big groups. Adding a constant λ to the effort of each member means that total group effort increases by λn_k. Therefore, bigger groups increase their effort more than smaller groups in absolute terms. But the standard Translation Invariance property implies that winning probabilities should remain invariant after that change. In order to correct this bias, we introduce the following axiom:

Axiom 7 (Group Translation Invariance) *For all* $\lambda > 0$ *and all* $k \in \mathbb{K}$

$$p_k(\mathbf{x}_1+\frac{\lambda}{n_1}\cdot\mathbf{1},..,\mathbf{x}_K+\frac{\lambda}{n_K}\cdot\mathbf{1}) = p_k(\mathbf{x}).$$

This property implies that if the total effort increases across all groups by the same positive amount λ by members increasing their effort by a fix amount $\frac{\lambda}{n_k}$, winning probabilities should remain constant. Group Translation Invariance thus levels the playfield: It eliminates the bias against big groups implicitly built in the standard Translation Invariance property, a bias which has been so far overlooked by the literature.[8]

Before stating our next theorem, consider the following definition:

Definition The impact function $h_k(x_k)$ is said to be *translatable* if

$$h_k(\mathbf{x}_k+\lambda\cdot\mathbf{1}) = h_k(\mathbf{x}_k)+\beta_k\lambda \qquad \textit{where } \beta_k, \lambda > 0.$$

We will refer to the scalar β_k as the degree of (linear) translatability of the impact function. Translatability is analogous to linear homogeneity when a fixed amount is added to the arguments of a function. We borrow this concept from the income distribution literature; it is a building block in the analysis of absolute inequality (Kolm, 1976a, 1976b; Blackorby and Donaldson, 1980).

We are finally ready to state our theorem characterizing the family of translation invariant difference-form CSFs.

Theorem 3 *If a CSF satisfies axioms A1-A4 and A6, then it satisfies (3) and the impact functions* $h_k(\mathbf{x}_k)$ *are translatable of the same degree* $\beta > 0$*. If A6 is replaced by A7, then each impact function* $h_k(\mathbf{x}_k)$ *is translatable of degree* βn_k*.*

Proof. A6 implies that $p_k(\mathbf{x}+\lambda\cdot\mathbf{1}) = p_k(\mathbf{x})$ so we can use the same reasoning as in the proof of Theorem 2 to establish that $\mathbb{K}^*$ does not change.

Now, combining Theorem 1 with the Translation Invariance axiom for any $k, l \leq K^*$ we obtain,

$$h_k(\mathbf{x}_k+\lambda\cdot\mathbf{1})-h_k(\mathbf{x}_k) = h_l(\mathbf{x}_l + \lambda\cdot\mathbf{1})-h_l(\mathbf{x}_l).$$

Since this holds for any $l, k \leq K^*$, the difference in impacts must depend only on λ so

$$h_k(\mathbf{x}_k+\lambda\cdot\mathbf{1})-h_k(\mathbf{x}_k) = \phi(\lambda),$$

[8] This bias against big groups is not built in the Scale Invariance property: When each member's effort increases in the same proportion, the total effort of all groups increases in that same proportion.

where $\phi(\cdot)$ is a continuous function because it is equal to the difference of two continuous functions. This expression holds for any $\lambda > 0$ so

$$h_k(\mathbf{x}_k+(\lambda+\mu)\cdot\mathbf{1})=h_k(\mathbf{x}_k+\lambda\cdot\mathbf{1})+\phi(\mu)=h_k(\mathbf{x}_k)+\phi(\lambda)+\phi(\mu),$$

implying that

$$\phi(\lambda+\mu)=\phi(\lambda)+\phi(\mu).$$

This is just the Cauchy functional equation whose only solution is of the form $\phi(\lambda) = \beta\lambda$ where $\beta > 0$ is an arbitrary real number.

The proof when A6 is replaced by A7 runs along the same lines. It must be that

$$h_k(\mathbf{x}_k+\frac{\lambda}{n_k}\cdot\mathbf{1})-h_k(\mathbf{x}_k)=\psi_k(\frac{\lambda}{n_k}),$$

where $\psi_k(\cdot)$ is also continuous because it is the difference of two continuous functions. Note that for this expression to hold true, it must be also that

$$\psi_k(\frac{\lambda}{n_k})=\psi_l(\frac{\lambda}{n_l}). \tag{8}$$

Because this holds for any $\lambda > 0$ then one can write

$$h_k(\mathbf{x}_k+\frac{\lambda+\mu}{n_k}\cdot\mathbf{1})=h_k(\mathbf{x}_k+\frac{\lambda}{n_k}\cdot\mathbf{1})+\psi_k(\frac{\mu}{n_k})=h_k(\mathbf{x}_k)+\psi_k(\frac{\lambda}{n_k})+\psi_k(\frac{\mu}{n_k}),$$

which implies that

$$\psi_k(\frac{\lambda+\mu}{n_k})=\psi_k(\frac{\lambda}{n_k})+\psi_k(\frac{\mu}{n_k}).$$

By induction, it is easy to see that this property implies that $\psi_k(\lambda) = n_k\psi_k(\frac{\lambda}{n_k})$. Hence, $\psi_k(\lambda+\mu)=\psi_k(\lambda)+\psi_k(\mu)$. This is again the Cauchy functional equation whose solution is $\psi_k(\lambda) = \beta_k\lambda$. This together with (8) implies that

$$\psi_k(\frac{\lambda}{n_k}) = \frac{\psi_k(\lambda)}{n_k} = \frac{\beta_k\lambda}{n_k} = \frac{\beta_l\lambda}{n_l} = \frac{\psi_l(\lambda)}{n_l} = \psi_l(\frac{\lambda}{n_l}),$$

so $\beta_k = \beta n_k$ where β is an arbitrary positive scalar. This completes the proof. ■

For an example of a translation invariant difference-form CSF, consider the following impact function which we employ in a companion paper (Cubel and Sanchez-Pages, 2014):

$$h_k(\mathbf{x}_k)=n_k^{\delta}\ln\left(\frac{1}{n_k}\sum_{i=1}^{n_k}e^{-\gamma x_{ik}}\right)^{-\frac{1}{\gamma}} \qquad \text{where } \gamma_k \geq 0 \text{ and } \delta \in \{0,1\}.$$

This is the natural logarithm of a CES function with exponential efforts. The parameter γ measures the degree of complementarity of members' efforts. This function is linear when $\gamma = 0$. It violates Monotonicity when $\gamma \to \infty$ as it converges to the weakest-link technology (Hirshleifer, 1983). This function satisfies Translation Invariance if $\delta = 0$ and Group Translation Invariance if $\delta = 1$.

One remark is in order at this point: In his Theorem 3, Münster (2009) characterizes the class of ratio-form CSF which are also translation invariant. He shows that for individual contests, this class boils down to the logistic CSF introduced by Hirshleifer (1989; 1991). In the literature, the logistic form is often referred to as a difference-form CSF. At the light of our axiomatization, we see this label as a misnomer. As our Theorem 1 shows, this form does not satisfy the Equalizing Consistency axiom. Hence, in order to be precise and rigorous, we believe that the logistic form should remain classified as a (translation invariant) element of the family of ratio-form CSFs.

4 Aggregation

So far, none of the properties we have posit on CSFs is specific to group contests. A distinctive feature of confrontations among groups is that members' efforts must be aggregated in some form. This is modelled through the impact function. Further assumptions on the aggregation of efforts are thus needed in order to obtain sharper characterizations. Consider the following axiom introduced by Münster (2009).

Axiom 8 (Summation) *For any $k \in \mathbb{K}$ consider two effort vectors $\mathbf{x}_k$ and $\mathbf{x}'_k$ such that $\sum_{i=1}^{n_k} x_{ik} = \sum_{i=1}^{n_k} x'_{ik}$. Then it must be that*

$$p_k(\mathbf{x}_k, \mathbf{x}_{-k}) = p_k(\mathbf{x}'_k, \mathbf{x}_{-k}).$$

This axiom implies that winning probabilities should remain invariant to changes in the distribution of efforts within groups which leave total group effort unchanged. In the context of lobbying or rent-seeking, where efforts are monetary, such assumption seems granted. Underlying this axiom is the assumption that efforts within groups are perfect substitutes, so the marginal productivity of individual effort does not depend on the effort made by other group members.

Let us now apply this property to our characterization of the family of difference-form CSFs.

Proposition 1 *If a CSF satisfies axioms A1-A5 and A8, then it is of the* *form (3) and the impact functions $h_k(\mathbf{x}_k)$ satisfy*

$$h_k(\mathbf{x}_k) = \alpha_k + \beta \ln(\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik}), \tag{9}$$

where α_k and *$\beta > 0$ are parameters.*

Proof. Given Theorem 2, we only need to prove that $g(\mathbf{x}_k) = \frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik}$. By A8, we know that the impact function can be expressed just as a function of the average effort of the group

$$h_k(\mathbf{x}_k) = h_k(\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik}, ..., \frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik}).$$

This together with expression (7), implies that it is possible to write $g(\mathbf{x}_k) = \phi(\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik})$. Since by Theorem 2 $\phi(\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik})$ must be homogeneous of degree one and because it is a function of one variable, it must be a multiple of a power function. Hence,

$$\phi(\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik}) = a\frac{1}{n_k}\sum_{i=1}^{n_k} x_{ik},$$

where by Theorem 2 again, $a = \phi(1) = g(\mathbf{1}) = 1$. This leads to the functional form (9) . Note that this function $g(\mathbf{x}_k)$ satisfies that $g(\mathbf{0}) = 0$. ■

The addition of Summation to our set of axioms produces a tighter characterization of the impact function. Proposition 1 highlights once more that the difference-form CSF can be scale invariant when the function mapping members' efforts into group impact is logarithmic. This result can also respond to a criticism often made against this family of CSFs and originally raised in Hirshleifer (2000)[9]: If the difference between the efforts of two contenders is kept fixed, the weaker side should be more likely to win as the absolute efforts of the contenders increase. More formally, $p_k^{\{k.j\}}(x_k, x_k + c)$ should be increasing in x_k, where $c > 0$. This property is called positive elasticity of augmentation by Hwang (2012). It is not satisfied in two-player contests by either the logistic ratio-form or the linear CSF introduced by Che

[9] "It might be thought a fatal objection against the difference form of the CSF that a force balance of 1,000 soldiers versus 999 implies the same outcome (in terms of relative success) as 3 soldiers versus 2! [...] Any reasonable provision for randomness would imply a higher likelihood of the weaker side winning the 1,000:999 comparison than in the 3:2 comparison." (Hirshleifer, 2000, p 779)

and Gale (2000). This is because these CSFs assume a linear mapping from effort to impact, which renders these functions translation invariant, that is

$$p_k^{\{k.j\}}(x_k, x_k + c) = p_k^{\{k.j\}}(x_k + t, x_k + t + c),$$

for $t > 0$. This in turn implies that $p_k^{\{k.j\}}(x_k, x_k + c)$ is constant in c so the elasticity of augmentation is zero. Such feature seems indeed unreasonable in many circumstances. It is easy to see that if the difference-form group CSF satisfies Scale Invariance and Summation, so the impact functions are as in (9), then for any $\lambda > 1$

$$\begin{aligned} p_k^{\{k,j\}}(\sum_{i=1}^{n_k} x_{ik}, c + \sum_{i=1}^{n_k} x_{ik}) &= p_k^{\{k,j\}}(\sum_{i=1}^{n_k} \lambda x_{ik}, \lambda c + \sum_{i=1}^{n_k} \lambda x_{ik}) \\ &\leq p_k^{\{k,j\}}(\sum_{i=1}^{n_k} \lambda x_{ik}, c + \sum_{i=1}^{n_k} \lambda x_{ik}), \end{aligned}$$

where the last inequality follows from the Monotonicity axiom. Therefore, the weaker group has a higher winning probability as the total efforts of the two groups increase whilst keeping the difference between total efforts constant; that is, the elasticity of augmentation is positive.

Let us now turn our attention to the case of translation invariant CSFs:

Proposition 2 *If a CSF satisfies axioms A1-A4, A6 and A8, then it is of the form (3) and the impact functions $h_k(\mathbf{x}_k)$ satisfy*

$$h_k(\mathbf{x}_k) = \alpha_k + \beta_k \sum_{i=1}^{n_k} x_{ik}, \tag{10}$$

where α_k and *$\beta_k > 0$ are parameters, and $\beta_k = \frac{\beta}{n_k}$ for all k. If A6 is replaced by A7, then $\beta_k = \beta$.*

Proof. By A8, it is possible to define the impact function as a function of $\sum_{i=1}^{n_k} x_{ik}$ so $h_k(\mathbf{x}_k) = \phi_k(\sum_{i=1}^{n_k} x_{ik})$. By Theorem 3 and A6, this function $\phi_k(\cdot)$ must satisfy

$$\phi_k(\lambda n_k + \sum_{i=1}^{n_k} x_{ik}) = \phi_k(\sum_{i=1}^{n_k} x_{ik}) + \beta\lambda.$$

Now define $H_k(t) = \exp\{\phi_k(t)\}$. Then

$$H_k(\lambda n_k + \sum_{i=1}^{n_k} x_{ik}) = \exp\{\beta\lambda\} H_k(\sum_{i=1}^{n_k} x_{ik}).$$

Let $\mathbf{x}_k = \mathbf{0}$. In that case, $H_k(\lambda n_k) = \exp\{\beta\lambda\} H_k(0)$. Substituting λn_k by t shows that it must be that

$$H_k(t) = a_k \exp\{\frac{\beta}{n_k} t\},$$

where $a_k = H_k(0) = \exp\{\phi_k(0)\}$.

Tracing back our steps,

$$\phi_k(\sum_{i=1}^{n_k} x_{ik}) = \ln H_k(\sum_{i=1}^{n_k} x_{ik}) = \ln a_k + \frac{\beta}{n_k} \sum_{i=1}^{n_k} x_{ik} = \alpha_k + \frac{\beta}{n_k} \sum_{i=1}^{n_k} x_{ik}$$

If we employ A7 instead, then the function $\phi_k(\cdot)$ must satisfy

$$\phi_k(\lambda n_k + \sum_{i=1}^{n_k} x_{ik}) = \phi_k(\sum_{i=1}^{n_k} x_{ik}) + \beta n_k \lambda.$$

Now define again $H_k(t) = \exp\{\phi_k(t)\}$ so

$$H_k(\lambda n_k + \sum_{i=1}^{n_k} x_{ik}) = \exp\{\beta n_k \lambda\} H_k(\sum_{i=1}^{n_k} x_{ik}).$$

Applying the same procedure, when $\mathbf{x}_k = \mathbf{0}$ it must be the case that $H_k(\lambda n_k) = \exp\{\beta n_k \lambda\} H_k(0)$, so substituting again

$$H_k(t) = a_k \exp\{\beta t\}$$

and

$$\phi_k(\sum_{i=1}^{n_k} x_{ik}) = \ln H_k(\sum_{i=1}^{n_k} x_{ik}) = \ln a_k + \beta \sum_{i=1}^{n_k} x_{ik} = \alpha_k + \beta \sum_{i=1}^{n_k} x_{ik}.$$

■

Summation plus Translation Invariance imply that impact functions must be linear. This has an additional consequence. Given that the form (3) is already separable in groups' impacts, any equilibrium of a contest with a translation invariant difference-form function must be in dominant strategies under two sets of circumstances: 1) when contenders are risk neutral, so the interpretation of $p_k(\mathbf{x})$ as a share or as a winning probability are equivalent; or 2) when individual utilities are non-linear and $p_k(\mathbf{x})$ is a winning probability. In these two cases, the marginal benefit of individual effort does not depend on the effort of any fellow group member or the effort of outsiders. It is thus natural that Beviá and Corchón (2014b) microfound this type of CSFs by means of dominant strategy implementation. Dominance solvability does not apply however when utilities are non-linear and $p_k(\mathbf{x})$ is instead a share of the prize contested, as in Levine and Smith (1995).[10]

One potentially undesirable consequence of the Summation axiom is that the resulting CSFs can admit biases. Take for instance the linear impact in (10) for the case of two-group contests, i.e. $K = 2$. In that case the difference-form (3) boils down to

$$p_k(\mathbf{x}) = \min\left\{\max\left\{\frac{1}{2} + \alpha_k - \alpha_l + \beta(\bar{x}_k - \bar{x}_l), 0\right\}, 1\right\},$$

where $\bar{x}_k$ denotes the average effort in group k. Note that group k has a head-start (handicap) whenever $\alpha_k > (<)\alpha_l$. The reason why the CSF admits this type of biases is because the Summation axiom remains silent on the relative success of different groups with the same total effort. One possibility is to modify the axiom in order to account for this problem.

Axiom 9 (Total Effort) *For any two groups $k, l \in \mathbb{K}$ such that $\sum_{i=1}^{n_k} x_{ik} = \sum_{i=1}^{n_l} x_{il}$ it must be that*

$$p_k(\mathbf{x}_k, \mathbf{x}_{-k}) = p_l(\mathbf{x}_l, \mathbf{x}_{-l}).$$

This axiom is a stronger version of Summation; it is actually a combination of Summation and the Between-Group Anonymity axiom in Münster (2009). It requires that two groups with the same total effort must have the same winning probability regardless of their size. Again, this property can make sense when efforts are monetary units, but not when efforts represent time or when group size matters. For instance, the impact of a group of 10 people demonstrating for 100 hours may not be the same as the impact of a group of 1000 people demonstrating for an hour.

The following Proposition shows that when Total Effort replaces Summation, the bias described above vanishes.

Proposition 3 *If A8 is replaced by A9, then the impact functions characterized in Propositions 1 and 2 must satisfy $\alpha_k = \alpha$ for all $k \in \mathbb{K}$.*

Proof. It suffices to show that when A9 holds, impact functions, whatever their functional form, should be identical across groups. To see this note that

$$h_k(\mathbf{x}_k) = h_k\left(\frac{\sum_{i=1}^{n_k} x_{ik}}{n_k}, ..., \frac{\sum_{i=1}^{n_k} x_{ik}}{n_k}\right),$$

[10] We thank Alberto Vesperoni for pointing this out.

because A9 also applies to changes in the distribution of efforts within groups which maintain total effort constant. Hence, for any vector $\mathbf{x}_k$ it is possible to write the impact of the group as a function of the total effort, i.e. $h_k(\mathbf{x}_k) = \phi_k(\sum_{i=1}^{n_k} x_{ik})$. Similarly for group l, that is, $h_l(\mathbf{x}_k) = \phi_l(\sum_{i=1}^{n_l} x_{il})$. From this it is clear to see that ϕ_k and ϕ_l are identical functions since by A9 they yield the same value whenever they are applied to the same argument. Hence, impact functions (9) and (10) must not differ across groups and $\alpha_k = \alpha$ for all $k \in \mathbb{K}$. ■

Total Effort eliminates biases in favor of certain groups. Such biases can be desirable in some instances. For instance, when they are the result of affirmative action policies aimed at fostering the participation of disadvantaged groups (Franke, 2012). In other contests, such as when a social planner seeks to commit to a fair and impartial sharing rule (Corchon and Dahm, 2011), these biases should be removed.

A particularly interesting CSF emerges when the Total Effort axiom and Scale Invariance are imposed: Denote by X_k the sum of efforts in group k and order groups by their total effort in a decreasing manner. Then, the CSF characterized by Proposition 3 must be

$$p_k(\mathbf{x}) = \begin{cases} \frac{1}{K^*} + \beta \ln \frac{X_k}{G_X} & \text{for } k \leq K^* \\ 0 & \text{otherwise} \end{cases}$$

where $G_X = (\prod_{l=1}^{K^*} X_l)^{\frac{1}{K^*}}$ is the geometric mean of groups' total efforts and K^* is the largest integer such that $\frac{X_{K^*}}{G_X} > \exp\{\frac{1}{\beta K^*}\}$.

5 Conclusion

In this paper, we have offered the first systematic study of group contests where winning probabilities depend on the difference between contestants' effective efforts. Our axiomatic characterization rested on an Equalizing Consistency axiom which imposes that the difference between winning probabilities in the grand contest and in the smaller contest must be identical across participants in the smaller contest. This contrasts with the consistency axiom employed in all the existing characterizations of the ratio-form CSF which posits that probabilities in any subcontest should be proportional to winning probabilities in the grand contest. One advantage of the Equalizing Consistency axiom is that it does not bound contestants to have a zero probability of winning a smaller contest when they have zero chances of winning a larger one.

The CSF resulting from our characterization generalizes the functional form introduced by Che and Gale (2000) and later employed by Rohner (2006), Besley and Persson (2008, 2009) and Gartzke and Rohner (2011). This functional form has three distinctive features: 1) Impacts across contestants are separable, 2) it awards a sure victory to a contender who overpowers its rivals by a large enough margin and 3) it allows contenders to enjoy a positive winning probability when their impact is zero provided that other contenders are not too strong.

We showed that, contrary to what it has been argued in the literature, difference-form CSFs can be homogeneous of degree zero, and that they do not force differences in winning probabilities to remain invariant when absolute differences in raw efforts remain constant, i.e. a zero elasticity of augmentation. In addition, we flagged-up that the Translation Invariance property builds in an implicit bias against big groups which should be corrected. In this process, we also argued that the logistic function (Hirshleifer, 1989, 1991), although often referred to as a difference-form CSF, does not actually belong to this family. In our opinion, this label should be reserved only to CSFs satisfying the Equalizing Consistency axiom, which the logistic form does *not* satisfy.

In the last part of the paper, we explored one possible technology of aggregation of efforts within groups. This helped us to sharpen our characterization of admissible impact functions. We also showed that a modified version of the Summation axiom in Münster (2009) can unbias the CSF, a desirable property in contexts where impartiality has a value.

The family of difference-form CSFs has not been employed in the contest literature as often as other functional forms. We hope that, by clarifying its properties, our axiomatization can persuade researchers in the area to include this family of CSFs in their toolkit. Of course, our characterization is normative and leaves out strategic interactions. Che and Gale (2000) showed that their linear difference-form CSF often leads to mixed-strategy equilibria and that any equilibrium in pure-strategies involves at most one contender exerting positive effort. One possible next step would be to explore whether the equilibria of contests under the generalized difference-form CSF axiomatized here still presents such features. In addition, this form implies the separability of contender groups, leading to dominant strategy equilibria when impacts are linear. We explore these issues in a companion paper (Cubel and Sanchez-Pages, 2014).

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