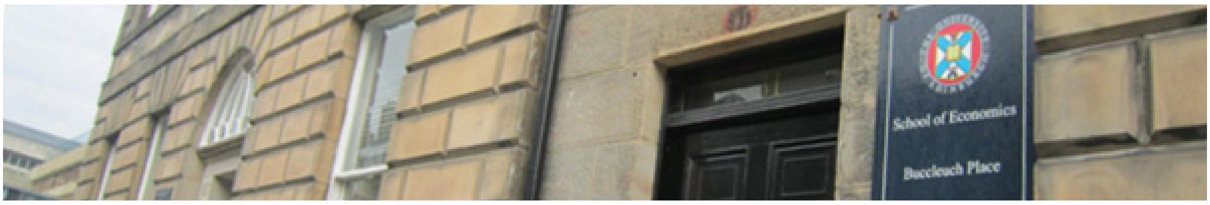

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Money Cycles

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Money Cycles*

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September 10, 2014

Abstract

Operating overheads are widespread and lead to concentrated bursts of activity. To transfer resources between active and idle spells, agents demand financial assets. Futures contracts and lotteries are unsuitable, as they have substantial overheads of their own. We show that money – under efficient monetary policy – is a liquid asset that leads to efficient allocations. Under all other policies, agents follow inefficient "money cycle" patterns of saving, activity, and inactivity. Agents spend their money too quickly – a "hot potato effect of inflation". We show that inflation can stimulate inefficiently high aggregate output.

1 Introduction

Operating overheads are widespread. Workers prepare for work by dressing professionally, travelling to work – often in peak traffic, re-familiarising themselves with their work plans (leaving family plans on hold), and checking in with colleagues. In manufacturing, significant engineering effort is applied to replace inventories with "just-in-time" production. But, when overheads can not be engineered away, activities are concentrated into bursts

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and batches. For example, oil tankers have high operating costs leading to large volumes being transported at a time, retailers restock with large deliveries, dry cleaners concentrate into batches due to the fixed cost of heating the drums, and income tax is evaluated every year. These bursts lead to demand for financial assets: holding large inventories typically involves trade credit, consignment, or futures contracts, and workers' rosters might be chosen by lottery. However, these assets all involve substantial overheads of their own. All of these assets involve the possibility of default, which is minimised with costly credit checks, collateral, and intermediation. Lotteries are particularly cumbersome: (i) to prevent default, money or some collateral would have to be posted before the winner is announced, and (ii) lotteries have a divisibility problem; for example five people are required if they would all like a 60% chance to work 100 hours (or win $100). However, there is one asset with very little overhead, namely *money*. Instead of buying from suppliers using trade credit, buyers could use money saved from previous sales to pay on delivery. Instead of random shift rosters, workers might decide to produce more on work days and save money to fund days off. Money is divisible, recognisable, and storable, giving it minimal transaction costs. Does this mean money is useful for overcoming overheads in productive activities?

To answer this question, we study a simple economy with a single productive activity. Output can be produced from labour, where an overhead of labour is required to begin production, and there is diminishing marginal productivity. Output is non-storable, and must be traded or consumed immediately. We compare the social planner's preferred allocations with equilibrium allocations in which the agents can hold and trade money.

Focusing our attention on a utilitarian social planner, we find that he prefers to allocate equal consumption to all agents. He prefers to make all work shifts the same length, but is indifferent between all feasible work shift allocations. This generalises the results by Rogerson (1988) and Prescott, Rogerson and Wallenius (2009). The key step is to show that the social planner's problem with overheads is equivalent to a convexified social planner's problem in which the production technology no longer has any overheads.

Can money achieve utilitarian-optimal allocations? We first study monetary equilibria when money is supplied with lump-sum taxation to support deflation at the rate of time preference, i.e. the Friedman rule. We find that all equilibria involve a "working class" and possibly a "leisure class" of agents. Working class agents consume the same amount every period, produce more than they consume in the long run, and may sometimes take a vacation. Members of the leisure class consume more than the working class and never work. Equilibria in which the leisure class is absent are utilitarian-optimal.

However, money is not perfectly frictionless, and is rarely supplied at the Friedman rule. How does money perform away from socially optimal monetary policy? We develop a theory that characterises all stationary symmetric monetary equilibria. We find that only equilibria approximating the utilitarian-optimal working class equilibria exist. In these equilibria, all agents proceed through a "money cycle" pattern of finite length of saving, production, and consumption. All aggregates are stationary – agents are at different phases of their cycle at any given moment. There is a "hot potato effect of inflation" which induces agents to spend their money too quickly on consumption, and hence render these money cycle equilibria inefficient.

We highlight some surprising features of our model, by refuting four conjectures that we argue are natural in the context of the monetary literature. First, we show that inflation need not depress economic activity, even in a model of complete information. Second, we show that even when consumption and leisure are normal goods, wealthier agents do not consume more and work less. Third, we show that Baumol-Tobin style Ss cycles are not the only possible money cycle structure. Finally, we show that symmetric equilibria need not exist, and two money cycles can co-exist in asymmetric equilibria.

Our work is related to three strands of literature. One literature is about the optimal allocation of labour when there are large setup costs. In the Rogerson (1988) model, labour is indivisible. He found that lotteries are efficient for convexifying the indivisibility. In Prescott *et al.* (2009), households choose their work intensity in continuous time when facing a non-concave production function. In this setting, equilibria with complete markets are utilitarian-optimal. Our work characterises *all* utilitarian-optimal allocations, shows that efficient monetary policy is equivalent to complete markets, and characterises equilibria under realistic monetary frictions.

Another literature is about financial frictions in converting financial assets into money. The only asset in our model is money, and this literature suggests a path forward for introducing other assets. Baumol (1952) and Tobin (1956) study a partial equilibrium model of money, when there is a fixed cost of liquidating an asset with high return. They show that an Ss policy is optimal (unlike in our setting). Alvarez, Atkeson and Kehoe (2002) also think about a fixed cost of liquidating financial assets, but assume that agents can not hold money across periods. Kaplan and Violante (2014) numerically explore a model with a fixed cost of liquidating high-return assets, and calculate the equilibrium response to fiscal stimulus payments. Their equilibrium calculations are complicated, but have a similar nature to our money cycle equilibria. Their agents proceed through Ss cycles involving a single liquidation, but we expect more complicated patterns akin to our money cycles would arise with an increasing marginal cost of liquidation.

Finally, our work is related to several monetary theories based on double-coincidence frictions in the tradition of Kiyotaki and Wright (1989). Faig (2008) and Menzio, Shi and Sun (2013) explore the use of money and lotteries to smooth out an overhead, namely sellers forego the opportunity to be buyers. In Faig (2008), sellers can only serve one buyer per period, so in equilibrium, workers never save for more than one vacation. Menzio *et al.* (2013) introduce firms to accommodate multilateral matching, so that workers save for several vacation periods. This leads the money distribution to be non-degenerate, and allows them to study how inflation affects people holding different amounts of money. They find that wealthier agents consume more. In contrast, we find that the poorest agents consume the most and that the wealth effect is non-monotonic.

The micro-founded monetary literature also includes several papers that explore whether inflation can stimulate economic activity. Ennis (2009) finds that inflation can increase search activity, but does not investigate the effect on productive activities; he also finds that agents front-load consumption. Nosal (2011) studies a stylised model in which buyers are sent to the back of the queue after trading, which leads to a perverse incentive to avoid trade. He finds that inflation is welfare-improving and increases aggregate output because the hot-potato effect offsets the trade-avoidance incentive. Other papers in this literature include Berentsen, Camera and Waller (2005) and Liu, Wang and Wright (2011).

This paper is organised as follows. Section 2 introduces a model in which agents have access to fiat money and incur overheads in economic activity. Section 3 characterises efficient allocations, with particular attention to utilitarian-optimal allocations. We then characterise the allocations that may arise under efficient monetary policy in Section 4. This monetary policy is implausible, so in Section 5 we characterise the equilibria that arise under inefficient monetary policy. We use this theory to address several natural conjectures that arise in various literatures in Section 6. Section 7 reflects on the contribution of the paper. Appendix A discusses a related model in which agents trade Lucas trees rather than money. Appendix B addresses some difficulties in calculating money cycle equilibria.

2 Environment

We construct a stationary general equilibrium model in infinite discrete time where agents discount at rate β. There is a continuum of agents $i \in [0, 1]$ who may produce any quantity $q_t \geq 0$ of a homogeneous consumption good at cost $c(q_t)$ in period t. We assume that the production cost $c(q)$ is strictly increasing, convex on $(0, \infty)$, and differentiable on $(0, \infty)$, but allow for a discontinuity at $q = 0$ which represents a fixed cost. Not producing anything is free, i.e. $c(0) = 0$. Agents receive utility $u(x_t)$ from consuming $x_t \geq 0$ units of this good. We assume that u is strictly increasing, differentiable, concave, and satisfies the Inada conditions $\lim_{x\to 0^+} u'(x) = \infty$, $\lim_{x\to 0^+} u(x) = -\infty$.

Throughout the paper, we assume that the consumption good is non-storable[1]. However, agents have access to fiat money, which they can trade for goods without incurring

[1] Non-monetary equilibria with *storable* goods are equivalent to monetary equilibria with zero inflation. Under deflation, agents would choose to hold money rather than store goods.

any fixed costs.[2] If an agent produces more than he consumes, he may sell the surplus for money on a spot market at the price p_t. Conversely, an agent can consume more than he produces buy trading at the same price p_t. Since other types of assets such as bonds, loans, and stocks involve transaction costs, agents would use money to smooth those costs out as well. For simplicity, we focus on just one fixed cost – in the production cost function – by excluding all non-monetary assets. Let M_t be the aggregate stock of money in nominal terms at the beginning of period t. We assume that it evolves over time depending on a lump sum transfer T_t issued by the government to all agents before starting to trade goods so that $M_{t+1} = M_t + T_t$. We write $1 + \pi_{t+1} = M_{t+1}/M_t$, where π is the money growth rate. Throughout the analysis, we take the government's monetary policy as exogenous so that M_t, π_t and T_t are exogenous variables.

If an agent holds m_t units of money at time t, his optimal choices satisfy the following Bellman equation

$$\begin{aligned} \tilde{V}_t(m) = &\max_{q\in\mathbb{R}_+,\, x\in\mathbb{R}_+,\, m'\in\mathbb{R}_+} u(x) - c(q) + \beta\tilde{V}_{t+1}(m') \\ &\text{s.t.} \quad p_t x_t + m_{t+1} = p_t q_t + m_t + T_t. \end{aligned} \tag{1}$$

We focus on stationary equilibria under a stationary monetary policy, so that the real transfer is stationary with $T = T_t/p_t$ and inflation is stationary with $\pi = \pi_t$. Note that T – unlike T_t – is endogenous because it is defined in terms of prices which are endogenous. Let $Z_t = M_t/p_t$ be the real value of the money stock. In a stationary equilibrium, the money growth rate $M_{t+1}/M_t = 1 + \pi$ coincides with the inflation rate p_{t+1}/p_t. Since $M_t + T_t = (1+\pi) M_t$, or in real terms, $Z_t + T = (1+\pi) Z_t$, the real money stock is stationary and can be expressed as $Z = T/\pi$. When we replace the nominal balances m_t with real balances $z_t = m_t/p_t$, the problem becomes stationary:

$$\begin{aligned} V(z) = &\max_{q\in\mathbb{R}_+, x\in\mathbb{R}_+, z'\in\mathbb{R}_+} u(x) - c(q) + \beta V(z') \\ &\text{s.t.} \quad x + (1+\pi) z' = q + z + T. \end{aligned} \tag{2}$$

We write the optimal production quantity as $q(z)$ and the optimal consumption policy as $x(z)$. The distribution of real money holdings is F. A *symmetric stationary equilibrium* in this environment is a tuple

$$[x(z), q(z), F(z), T]$$

such that

[2] In Appendix A, we explore how our analysis would change if agents could trade Lucas trees instead of money. We focus on money in the body of the paper because money has very small overheads.

- the policies $q(z)$ and $x(z)$ solve the stationary problem (2) given T;
- goods and money markets clear so that supply equals demand:

$$\int q(z)\ dF(z) = \int x(z)\ dF(z) \quad \text{and} \quad \frac{T}{\pi} = \int z\, dF(z)\,;$$

- the distribution of money holdings is stationary:

$$F(z') = \int I\left\{[q(z) - x(z) + z + T]/(1+\pi) \leq z'\right\}\ dF(z)$$

  which means that the measure of agents holding less than or equal to z' real balances of money at the beginning of the next period has to equal the measure that saved z' or less by working, consuming, and getting lump sum transfers.

3 Efficient Allocations

We characterise the efficient allocations of consumption and production when there are fixed costs of production. In this section, we focus most of our attention on a utilitarian social planner that puts equal welfare weight on all agents. We find that this planner allocates each agent a mix of "full-time" shifts and vacations. He allocates each agent the same consumption, but does not attempt to allocate shifts evenly – he is indifferent among all feasible shift allocations.

Our proof technique is to introduce a second production cost function, namely the convex hull of the economy's production cost (see Figure 1). We find that a general class of social planners' values are the same, regardless of which production cost function he faces. We then solve the utilitarian social planner's convexified problem to characterise utilitarian-optimal allocations.

The social planner prioritises the utility of agent $i \in [0,1]$ according to a Negishi weight, η_i. The social planner calculates social welfare as

$$W(\eta) = \max_{\{x_{it}, q_{it}\}} \int_{[0,1]} \eta_i \sum_{t=0}^{\infty} \beta^t [u(x_{it}) - c(q_{it})] di \tag{3}$$

$$\text{s.t.} \int_{[0,1]} [x_{it} - q_{it}] di = 0 \text{ for all } t. \tag{4}$$

Our approach is to study an equivalent social planner's problem that is easier to solve. We will establish that the (non-convex) production cost function, $c(\cdot)$ has a range of redundant output levels, $(0, \bar{q})$ that the social planner would never wish to allocate to any worker.

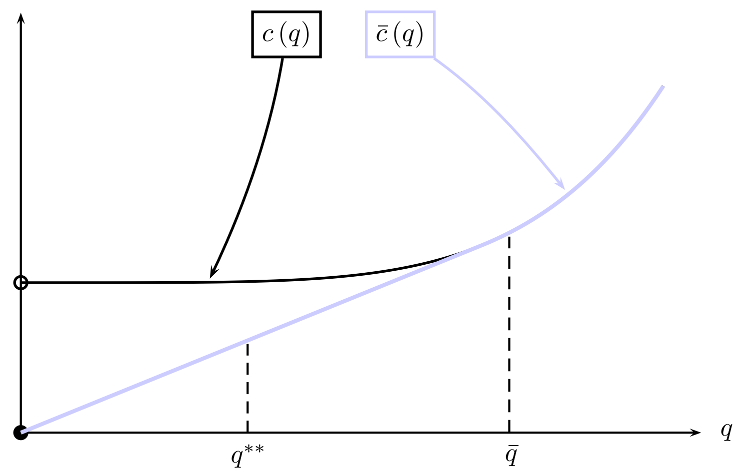

Figure 1: The convex hull of the cost function

We reduce the costs of these redundant outputs as far as we can while still keeping them redundant, i.e. until the planner is indifferent between allocating them to some workers or not. The resulting production cost function is the *convex hull*, denoted $\bar{c}(\cdot)$, and is depicted in Figure 1. Formally speaking, it is the upper envelope of all affine functions that lie below $c(\cdot)$. The convex hull is linear on $(0, \bar{q})$, and coincides with the original cost function elsewhere.[3]

The first part of Theorem 1 verifies that indeed, the social planner has the same value under either production cost function, precisely because they only differ on redundant choices $(0, \bar{q})$. The convexified problem is simpler, because there is no longer any need to treat agents differently to avoid the inefficient output levels $(0, \bar{q})$. Therefore, the remaining parts of the theorem focus on utilitarian social welfare that allocates equal weight to each agent. Part (ii) finds that autarky is a utilitarian-optimal allocation, where agents consume and produce the x^{**} that solves

$$u'(x) = \bar{c}'(x). \tag{5}$$

But this is not the only utilitarian-optimal allocation. The linear section of $\bar{c}(\cdot)$ also leads uneven workloads to be utilitarian-optimal.

Part (iii) uses this characterisation of the convexified problem to understand the original social planner's problem. While the autarky allocation is optimal in the convexified

[3] Note that when $c(\cdot)$ has constant marginal cost, it never intersects its convex hull $\bar{c}(\cdot)$ above $q = 0$. We focus on economies in which the marginal cost increases to infinity.

problem, we know that it is inefficient in the original problem. Nevertheless, since the social welfare in both problems is equal, it follows that there must be some way to rearrange production to achieve the same level of welfare in the original problem.

Part (iv) calculates utilitarian welfare by using the autarky allocation.

Theorem 1. *(i) The social planner's problem and the convexified social planner's problem coincide, i.e.* $W(\eta) = \bar{W}(\eta)$ *for all* η*.*

For the remaining parts, suppose $x^{**} \leq \bar{q}$ *and* $\eta = 1$*, i.e.* $\eta_i = 1$ *for all* i*.*

(ii) All solutions to the convexified social planner's problem involve $x^*_{it} = x^{**}$ *and* $q^*_{it} \in [0, \bar{q}]$ *for almost all* (i, t)*.*

(iii) All solutions to the social planner's problem involve $x^*_{it} = x^{**}$ *and* $q^*_{it} \in \{0, \bar{q}\}$ *for almost all* (i, t)*.*

(iv) $W(1) = \bar{W}(1) = \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta}$.

Proof. (i) We first note that $W(\eta) \leq \bar{W}(\eta)$, because $\bar{c}(q) \leq c(q)$ for all q, i.e. the convexified problem is a relaxation of the original problem.

To show that $W(\eta) \geq \bar{W}(\eta)$, we first simplify both functions. The additively separable nature of the social planner's problem means that it can be reformulated recursively as

$$W(\eta) = \max_{\{x_i, q_i\}} \int_{[0,1]} \eta_i [u(x_i) - c(q_i)] di + \beta W(\eta) \tag{6}$$

$$\text{s.t.} \int_{[0,1]} [x_i - q_i] di = 0. \tag{7}$$

This means the social planner's problem is essentially a static problem (unlike the agent's problem in the monetary economy). A similar reformulation is possible for the convexified social planner's problem.

To show that $W(\eta) \geq \bar{W}(\eta)$, we show that any solution $\{x^*_i, q^*_i\}$ to the later can be transformed into a solution of the former without any loss of social welfare. Now, let $A = \{i : q^*_i \in (0, \bar{q})\}$ be the set of agents who work strictly between 0 and $\bar{q}$ hours. We will exploit the linearity of the convexified cost function $\bar{c}(\cdot)$ on $[0, \bar{q}]$ to rearrange these workers' hours to the boundary, $\{0, \bar{q}\}$, where $c(\cdot)$ and $\bar{c}(\cdot)$ coincide. If the (Lebesgue) measure of these agents $\lambda(A)$ is zero, then no rearrangement is necessary. Suppose then that $\lambda(A) > 0$. In this case, η_i must be the same for almost all workers in A. (Otherwise, the social planner would strictly prefer to reallocate work from workers with high Negishi weights to those with low Negishi weights, violating the optimality of $\{x_i^*, q_i^*\}$.) Since η_i is the same number for almost all agents in A, the social planner is indifferent about reallocating work to the boundaries, 0 and $\bar{q}$. Therefore, this reallocation has the same value in both social planner problems, and implies $W(\eta) \geq \bar{W}(\eta)$.

(ii) Now suppose that $\{x_i^*, q_i^*\}$ is a solution to the convexified problem in which $\eta_i = 1$ for all i. We consider two possible reallocations the social planner might make: (a) instructing agent i to give ϵ of his consumption up in favour of agent j, and (b) instructing agent i to change his production and consumption by the same amount, ϵ. Since these reallocations involving only a finite number of agents, they do not affect social welfare. However, the per-agent objectives

$$\eta_i u(x_i^* - \epsilon) + \eta_j u(x_j^* + \epsilon)$$
$$\eta_i [u(x_i^* + \epsilon) - \bar{c}(q_i^* + \epsilon)]$$

are maximised by $\epsilon = 0$ for almost all agents (or pairs of agents in the first case). Since the Negishi weights are equal, we obtain the first-order conditions,

$$u'(x_i^*) = u'(x_j^*)$$
$$u'(x_i^*) = \bar{c}'(q_i^*) \text{ if } q_i^* > 0.$$

The first equality implies that almost all agents consume the same amount, which we denote x^*. The second equality implies that either almost all produce the same amount $q^* > \bar{q}$ or almost all agents produce in the range $[0, \bar{q}]$. The former case is impossible, as the resource constraint would imply $x^* = q^* = x^{**}$, violating the condition that $x^{**} < \bar{q}$. Therefore, $u'(x^*) = \bar{c}'(\bar{q})$, which implies $x^* = x^{**}$.

Therefore, almost all agents consume x^{**} and produce in $[0, \bar{q}]$ in solutions to the convexified social planners problem when $x^{**} < \bar{q}$ and all agents are equally weighted.

(iii) This follows from parts (i) and (ii).

(iv) Part (ii) found the autarky allocation involving consumption and production of x^{**} is utilitarian-optimal in the convexified problem. Therefore, utilitarian welfare is

$$W(1) = \bar{W}(1) = \int_{[0,1]} \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta} \, di = \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta}. \tag{8}$$

□

Prescott *et al.* (2009, Sections 2 and 3) study the social planner's problem in a related environment with a non-concave production function. Our analysis is a somewhat more complicated, because we accommodate agents that discount that future and because we accommodate non-utilitarian social planners. Apart from that, they assume time is continuous, where each instant is interpreted as one week, and that agents have finite lifetimes. The main conclusions are the same: the utilitarian social planner has all agents consume the same amount at all times, and work the same hours as each other (during work weeks). Their paper in turn builds on the work of Rogerson (1988), in which workers live for one period and make a single indivisible labour choice. That paper found that lotteries are efficient. Lotteries would also be utilitarian-optimal in our economy and in the economy of Prescott *et al.* (2009), if they were available.

4 Efficient Saving with Money

Real-world market institutions involve a wide range of frictions that leaves a gap between efficient and equilibrium allocations. As discussed in the introduction, we focus on monetary institutions, rather than credit, lotteries, bonds, or capital, which involve substantial fixed costs. This section argues that an idealised form of monetary institutions (with an unrealistic monetary policy) leads to utilitarian-optimal allocations. This result will help us understand the frictions of more realistic monetary policies in the next section.

The previous section established that utilitarian-optimal allocations involve constant consumption, a constant work shift length, and vacations. This means agents need a way to save for vacations. Money is one such institution. Monetary policy – specifically, the return of holding money $(1 + \pi)$ – determines the agent's incentive to save and smooth out consumption. We need a policy that makes the agent reject deviating from stationary consumption by saving an extra unit of money. This is the case when the return of holding money equals the discount rate, i.e. the Friedman rule, $1 + \pi = \beta$. We find that a large class of equilibria are possible, but argue that only the utilitarian-optimal equilibria are robust.

Our study of the agent's problem mirrors that of the social planner's problem in the previous section. In Theorem 2, we establish that the agent's value of holding money is the same, regardless of whether the agent faces production cost $c(\cdot)$ or the convexified cost function $\bar{c}(\cdot)$. That allows us to show that agents either take vacations or work a full shift, when they are tight on money (we call them "working class"), or they never work and consume their return on savings (we call these agents "leisure class"). We will argue below that only equilibria without any leisure class are robust. Part (iv) establishes that these robust equilibria are utilitarian-optimal.

Theorem 2. *Let $\bar{V}$ be the agent's convexified value function. Suppose that $x^{**} \leq \bar{q}$. For every real value of transfers T at the Friedman rule, there is a cut-off $\bar{z} = (x^{**} - T)/(1-\beta)$ such that:*

(i) The agent's actual and convexified value functions coincide, with

$$V(z) = \bar{V}(z) = \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta} + c'(\bar{q})(z - Z) \tag{9}$$

for all $z \in [0, \bar{z}]$, where $Z = T/\pi$,

(ii) agents with $z < \bar{z}$ ("working class") consume x^{**} every period, and only ever produce 0 or $\bar{q}$,

(iii) agents with $z > \bar{z}$ ("leisure class") consume $z(1-\beta) - T > x^{**}$ every period and never work, and

(iv) all symmetric stationary equilibria $[x(\cdot), q(\cdot), F(\cdot), T]$ without any leisure class are utilitarian-optimal, i.e.

$$\int V(z)\, dF(z) = W(1) = \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta}.$$

Proof. (i) We guess-and-verify that $\bar{V}$ as defined in (9) is a fixed-point of the agent's convexified problem's Bellman operator,

$$\bar{\Phi}(V)(z) = \max_{x,q} u(x) - \bar{c}(q) + \beta V\left(\frac{z+q+T-x}{1+\pi}\right)$$
$$\text{s.t. } x \leq z + q + T.$$

Technically speaking, we only claim this is the formula for $\bar{V}$ on a subset of its domain. If the agent has a large amount of savings (above $x^{**}/(1-\beta) + Z$), he can afford to buy x^{**} and pay the lump-sum tax $-T$ from the return and never work again; in this case, the marginal value of z has curvature, so the linear formula overestimates $\bar{V}$. Therefore, it suffices to check that the relevant portion of $\bar{V}$ is unchanged by the Bellman operator, despite $\bar{V}(z)$ being too large for large z. Applying the Bellman operator to $\bar{V}$ under the Friedman rule $1 + \pi = \beta$ gives

$$\bar{\Phi}(\bar{V})(z) = \max_{x,q} u(x) - \bar{c}(q) + c'(\bar{q})(z + q + T - x - \beta Z) + \beta\frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta}$$
$$\text{s.t. } x \leq z + q + T.$$

First-order conditions give $x = x^{**}$ and the agent being indifferent between choosing any feasible $q \in [0, \bar{q}]$.[4] For simplicity, we choose $q = x = x^{**}$:

$$\begin{aligned}\bar{\Phi}(\bar{V})(z) &= u(x^{**}) - \bar{c}(x^{**}) + c'(\bar{q})(z + T - \beta Z) + \beta \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta} \\ &= \frac{u(x^{**}) - \bar{c}(x^{**})}{1-\beta} + c'(\bar{q})(z - Z) \\ &= \bar{V}(z).\end{aligned}$$

Thus, we have verified the formula (9) for $\bar{V}$, and that there exists an optimal policy involving $q(z) \in \{0, \bar{q}\}$ for all z.

We now establish that $\bar{V} = V$. Since $\bar{V}$ is the value of a relaxed problem, we know that $\bar{V}(z) \geq V(z)$ for all z. Since an optimal solution of the relaxed problem is feasible in the original problem, we know that $\bar{V}(z) \leq V(z)$. We conclude that $\bar{V} = V$.

(ii) First, we claim that only $q \in \{0, \bar{q}\}$ can be optimal choices. Suppose for the sake of contradiction, that $\hat{q} \in (0, \bar{q})$ were optimal at $\hat{z} < \bar{z}$. This would imply $c(\hat{q}) > \bar{c}(\hat{q})$, and hence $\bar{V}(\hat{z}) > \bar{V}(\hat{z})$, which violates (i).

Second, the optimal consumption choice in both problems is the same, because in both cases, the first-order conditions imply $u'(x) = c'(\bar{q})$.

(iii) If $z \geq \bar{z}$, the agent may wish to choose $z' > \bar{z}$, so (9) does not apply. Nevertheless, the Euler equation $u'(x) = \beta/(1+\pi)u'(x') = u'(x')$ holds, so the agent chooses constant consumption. (See Lemma 1 below.) The agent can afford to consume at least $-\pi\bar{z} + T = x^{**}$, so the marginal utility of consumption $u'(x^{**})$ is lower than the marginal cost of working, $c'(\bar{q})$, so this agent never works. The agent never transitions to become working class – otherwise, his consumption would drop to x^{**}, violating the Euler equation.

(iv) Follows from (9).

□

It is well-known that the Friedman rule monetary policy often has a large class of equilibria that do not survive when the monetary policy is changed slightly. We argue that equilibria involving a leisure class are exotic in this sense. The corollary below establishes that leisure class equilibria do exist at the Friedman rule, but we establish in the next section that they do not exist for any other monetary policy. This means only utilitarian-optimal equilibria have any counterpart away from the Friedman rule.

[4] It follows from $x^{**} \leq \bar{q}$ that the agent chooses $q \leq \bar{q}$.

The simplest leisure class equilibrium involves the leisure class extracting all the surplus from the working class. In this "exploitation" equilibrium, the working class agents work every day, while the leisure class agents never work and live off the return from holding money.

Corollary 4.1. *If $x^{**} \leq \bar{q}$, then for any measure of working class agents, $e \in [x^{**}/\bar{q}, 1)$, there exists an ("exploitation") equilibrium with the following properties:*

(i) Lump-sum transfers are $T = -(\bar{q} - x^{**})$.

(ii) Working class agents never take vacations and hold $z = 0$. (They work $\bar{q}$ and consume x^{**} every period, as before.)

(iii) Leisure class agents consume $x^l = \frac{e}{1-e}\bar{q}$ every period and hold $z = \frac{x^l - T}{\beta}$. (They never work, as before.)

Proof. It is straightforward to verify that under the proposed consumption and work policies, the conjectured distribution of money holdings is stationary.

By Theorem 2 part (ii), the agents holding money $z = 0$ are working class, and prefer to consume x^{**} every period and to work $q \in \{0, \bar{q}\}$. Since only $q = \bar{q}$ is feasible at $z = 0$, we conclude the working class choose to work every period.

When the size of the working class e is sufficiently large, the agents holding $z = \frac{x^l - T}{\beta}$ are above the leisure class cut-off of Theorem 2 part (iii). According to the theorem, these agents prefer to consume x^l every period and never work.

The quantity x^l was chosen so that the goods market clears. Walras' law then implies that the money market also clears. □

Prescott *et al.* (2009) also consider equilibria that implement efficient allocations. They study an Arrow-Debreu economy in which agents can effectively borrow and save at the frictionless interest rate. In the monetary economy we study, agents can not hold negative money balances, and are therefore credit constrained. This means that every utilitarian-optimal monetary equilibrium at the Friedman rule is an equilibrium in their economy, but not vice versa. Agents in their economy are only endowed with time (and not money), so no leisure class equilibrium can arise.

5 Inefficient Money Cycles

The previous section studied an idealised version of monetary institutions, in which the value of money deflates at precisely the rate of time preferences. Such a monetary policy is difficult to implement, as it requires subsidising money holders, and doing so at a knife-edge rate. Even slightly over-subsidising money leads to fundamental non-existence problems (agents want to hold too much money, and the money market does not clear). Therefore, we study the more realistic situation in which money is under-subsidised, i.e. with inflation or less deflation than the Friedman rule.

In Theorem 3, we establish that away from the Friedman rule, agents' decisions follow finite money cycles that begin with no money holdings in every stationary equilibrium. We show that money can only have value for money cycles of length two or more. In a money cycle equilibrium, the distribution of money holdings is non-trivial, with a finite support. In Theorem 4, we characterize the choices throughout each cycle. First, agents front-load consumption in response to the inflation tax. We interpret this as a hot potato effect of inflation. Second, agents also attempt to back-load production, but this is limited by the no-credit constraint and their preference to front-load consumption. Third, money cycles begin with work in the first period and end with a vacation in the last period. In Corollary 5.3, we show that when agents face a constant marginal production cost, money cycles have a monotonic Baumol-Tobin structure with only one work day. However, we provide an example with increasing marginal cost in which the money holdings are not monotonically decreasing throughout the money cycle. Figure 2 shows an example of a simple money cycle.

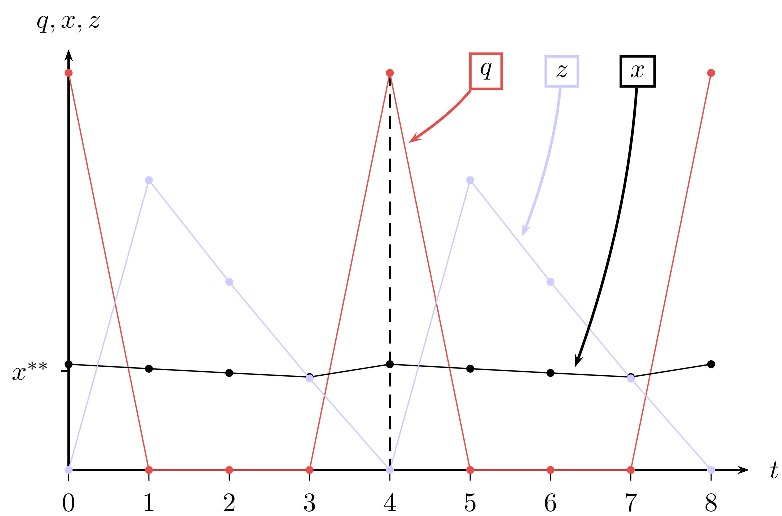

Figure 2: A simple money cycle. The symbols q, z and x denote production, real balances of money holdings, and consumption respectively.

The Euler equation is important for characterizing the equilibrium.

Lemma 1. *If x, x', and $z' > 0$ are optimal choices at some money balance z, then then*

$$u'(x) = \frac{\beta}{1+\pi} u'(x'). \tag{10}$$

Proof. When $z' > 0$, the agent can change today's consumption x to $x+\epsilon$ and tomorrow's consumption x' to $x' - (1+\pi)\epsilon$ without violating any no-credit constraints, for ϵ sufficiently close to zero. The agent's lifetime value changes by

$$[u(x+\epsilon) - u(x)] + \beta[u(x' - (1+\pi)\epsilon) - u(x')].$$

Since the agent rejects any such deviation, $\epsilon = 0$ maximises this value change, so the first-order condition (10) holds. $\square$

The first-order condition

$$u'(x) = c'(q) \tag{11}$$

applies on work days (when $q > 0$).

Definition 1. *We say that an agent's decisions $(\{q_t\}, \{x_t\}, \{z_t\})$ follow a money cycle of length $n > 0$ if n is the smallest number such that $z_t = z_{t+n}$ for all t. We say that the money cycle is non-trivial if $n > 1$.*

Theorem 3. *In every symmetric stationary monetary equilibrium away from the Friedman rule (i.e. for $1+\pi > \beta$), agents' decisions follow a (possibly trivial) money cycle that contains 0 money holdings. Every agent cycles through the same sequence.*

Proof. Suppose $(\{q_t^*\}, \{x_t^*\}, \{z_t^*\})$ is an optimal solution to the agents' problem. We argue below that $\{z_t^*\}$ includes 0 for some t. By truncating the start of the sequences, we repeat the argument to conclude that z_t^* includes a second 0. In a stationary equilibrium, the same decisions are taken whenever $z_t^* = 0$. We conclude that the entire sequence of decisions between the first and second time $z_t^* = 0$ is repeated over and over to form a money cycle. Since every agent's sequence of money holdings z_t^* includes 0 and all agents follow the same policies in symmetric equilibria, it follows that every agent follows the same sequence of decisions. Therefore, it suffices to show that $\{z_t^*\}$ includes 0 for some t.

For the sake of contradiction, suppose that $z_t^* > 0$ for all t. In this case, the Euler equation (10) applies every period so that

$$u'(x_1^*) = \frac{\beta}{1+\pi} u'(x_2^*) = \cdots = \left(\frac{\beta}{1+\pi}\right)^{t-1} u'(x_t^*). \tag{12}$$

When the money growth rate is above the Friedman rule (i.e. $1+\pi > \beta$), the first term on the right side converges to 0, so $u'(x_t^*) \to \infty$ and hence $x_t^* \to 0$. Due to the Inada condition, this implies $u(x_t^*) \to -\infty$ and hence $V(z_t^*) \to -\infty$. However $V(\cdot)$ is bounded below by the autarky payoff of

$$\max_x \frac{1}{1-\beta}\left[u(x) - c(x)\right].$$

This contradiction implies that the premise that $z_t^* > 0$ for all t is false. $\square$

Note that there may be multiple stationary equilibria. In this case, it would also be an equilibrium for agents to switch from one money cycle to another. However, such an equilibrium is not stationary.

The following corollary shows that a trivial money cycle (in which agents always hold zero real-balances of money) can not be a monetary equilibrium. Rather, trivial money cycles give rise to autarky.

Corollary 5.1. *In every trivial money cycle equilibrium (with length one), money has no value.*

Proof. As shown above, every money cycle includes zero. This implies that each agent's real balance is $z*_t = 0$ in every period t. This means that the real value of the money stock M_t is 0, which is only possible if money has no value (i.e. $p_t = \infty$). $\square$

Next, we show that the distribution of money holdings is non-degenerate, but has a simple structure. This is because agents trade positions with each other in the money cycle.

Corollary 5.2. *In every symmetric stationary equilibrium, the distribution F of (real balances of) money holdings has equal mass over a finite set.*

Proof. The support of the distribution of real balances coincides with the equilibrium sequence of real balances. Since each agent cycles through the sequence at the same pace, the measure of agents at each point of the sequence is equal, so the stationary distribution has equal mass at each point in its support. $\square$

Without loss of generality, we say that the start of the money cycle is when agents hold no money. The following theorem summarizes the properties of money cycles.

Theorem 4. *In every stationary equilibrium away from the Friedman rule (i.e. $1 + \pi < \beta$), agents proceed through money cycles that*

(i) have decreasing consumption, with marginal utility increasing in proportion to the inflation tax, $(1 + \pi)/\beta$.

(ii) the subsequence of non-zero production quantities is increasing throughout the money cycle, with (shadow) marginal cost increasing in proportion to the inflation tax.

(iii) begin with work and end with vacation.

Proof. (i) Previously, we found the Euler equation (10) holds between period t and $t+1$ whenever $z_{t+1}^* > 0$. Since z_{t+1}^* is greater than 0 in every period before the end of a money cycle, the Euler equation holds between every period within a money cycle. The Euler equation implies consumption decreases with marginal utility increasing in proportion to $(1+\pi)/\beta$.

(ii) Follows from part (i) and the production first-order condition (11).

(iii) Since the agent begins a money cycle with no money, it must work to finance its first period consumption (which is the highest level of consumption in the cycle by part (i) and hence can not be financed by transfers).

Suppose the agent works in the last period. Since production is greater or equal to consumption in the first period, parts (i) and (ii) imply that production is strictly greater than consumption in the last period. This contradicts the conclusion that savings are 0 in the last period.

□

The following corollary shows that when agents face a constant marginal production cost, a Baumol-Tobin style work-vacation pattern such as the example in Figure 2 emerges endogenously.

Corollary 5.3. *If $c(q)$ is affine on $(0,\infty)$, then money cycles contain only one work day.*

Proof. Since $c'(q_t) = u'(x_t)$ on work days, and $u'(x_t)$ increases over the money cycle, it follows that $c'(q_t)$ must increase on work days. But when $c(q)$ is affine, $c'(q)$ is a constant. □

However, if the marginal cost is strictly increasing, money cycles can become more complex. We explore complex cycles in the next section.

Prescott *et al.* (2009) only study equilibria in which agents have access to perfect credit markets. If agents only had access to money supplied away from the Friedman rule, then we would expect money cycles would arise: consumption and work would fluctuate inefficiently due to the hot-potato effect, and agents' work weeks would be too short. On the other hand, households often have access to other assets like retirement plans, houses, and mortgages that have a higher return than money. Adjusting financial portfolios involve substantial overheads, so households would use money for short-term smoothing and illiquid assets for long-term smoothing, as in Alvarez *et al.* (2002) and Kaplan and Violante (2014). In this paper, we focus on a single overhead at a time for simplicity. In future work, we think it would be interesting to combine the two types of overheads to understand retirement and work-week allocations arise.

Simple money cycles also arise in Baumol (1952), Tobin (1956), and Menzio *et al.* (2013). We illustrate the differences between these money cycles in the following section.

6 Refuted Conjectures

In this section, we refute four natural conjectures.

Conjecture 1: Inflation depresses output and trade

Keynes (1923, p. 45–) suggested that the opposite ought to be true – that inflation might stimulate trade because idle money loses value. Lucas (1996) discusses the conjecture thoroughly and points out that in standard micro-founded monetary models it is true:[5] the Friedman rule typically maximizes welfare, and inflation reduces production, consumption, and trade. However, Lucas (1996) also cites many models that suggest the opposite, but "any of these models leads to the distinction of anticipated and unanticipated changes in money […] None of these models deduces the [opposite of Conjecture 1] from assumptions on technology and preferences alone." In our model, this conjecture is true for many parameter values. For example in Figure 2, aggregate consumption and production are below the Friedman rule level, x^{**} (see Theorem 2). However, in Figure 3, inflation stimulates aggregate consumption and production above the efficient level. In this example, the fixed cost is high, so that $\bar{q}$ is very large compared to x^{**}, which means the agent would ideally work very infrequently. However, inflation is also high, which in contrast makes the agent want to have a short money cycle, i.e. work frequently. As a result, the agent works inefficiently often, and because the marginal cost of production is still relatively low, he doesn't reduce his per-shift hours by much. Therefore, inflation stimulates aggregate production above the efficient level, and does so without any monetary surprises or asymmetric information.

Note that stimulus is distinct from the hot-potato effect. Ennis (2009) was motivated to study the hot-potato effect in order to find a mechanism by which inflation might stimulate output. In his model, the hot-potato effect causes inflation to increase the number of transactions, but also to reduce transaction sizes. He does not establish whether the net effect of inflation can ever stimulate aggregate output.

[5] Earlier in standard overlapping generations models, later in search models such as Lagos and Wright (2005) and many others.

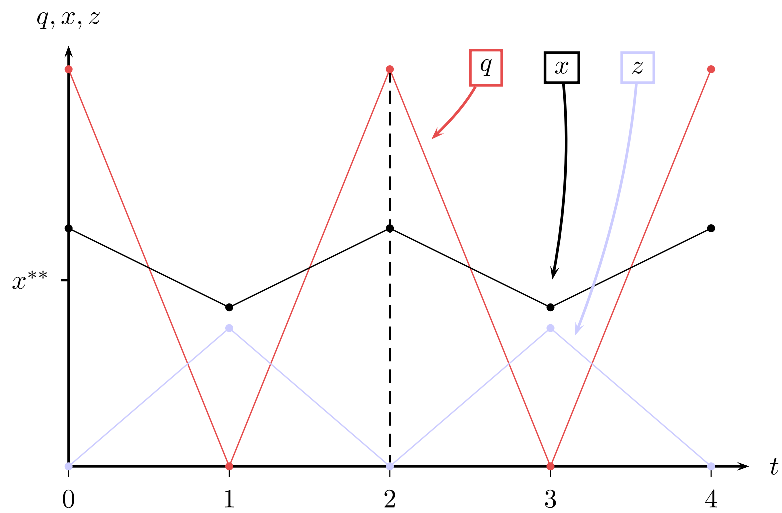

Figure 3: A money cycle with over-production and over-consumption, with $u(x) = x^{0.3}$, $c(q) = 0.15 + q^{1.3}$, $\beta = 0.98$ and $\pi = 0.3$.

Conjecture 2: Wealthier agents consume more and produce less

It is natural to expect if consumption and leisure are normal goods, then agents holding more money would choose more of both. However, Theorem 3 establishes that this is not true in any money cycle equilibrium: the poorest agents are those at the start of the cycle without any money. Yet these agents consume the most, and produce the least (other than those on vacation).

In most of the micro-founded monetary literature, the models are too simple to address this conjecture because all agents hold the same amount of money. But in other models with a non-trivial distribution of money, the conjecture holds true. In Menzio *et al.* (2013), agents have a discrete choice of whether to be a buyer or seller, which they can smooth out using lotteries.[6] Since lotteries incur no overheads, the value functions are concave, and the policy functions are monotonic. Therefore, this conjecture can only fail in a model with overheads. It is therefore no surprise that in other models with overheads, this conjecture also fails. In a model of overheads in asset markets, Kaplan and Violante (2014, fig 1) depict an equilibrium with non-monotonic consumption.

[6] The use of lotteries to convexify out the buy/sell choice was first developed by Faig (2008).

Conjecture 3: Money cycles consist of a work spell followed by a vacation spell

In Baumol-Tobin, agents are only active in the first period of their cycle. It's tempting to conjecture that agents might work several times (due to increasing marginal cost) until they can afford a vacation at the end of their cycle. But Figure 4 shows that much more complicated arrangements can be optimal. A single money cycle contains two work spells and two vacation spells.

In Menzio *et al.* (2013), the agents follow a stochastic version of Baumol-Tobin money cycle that satisfies the conjecture. Agents start their cycle with no money, work once, and do not work again until they run out of money. When agents run low on money, they play lotteries rather than working. If lotteries were unavailable, then agents would fall back on using money to convexify their problem; we expect that for some parameter values, they would work in multiple spells throughout the cycle.

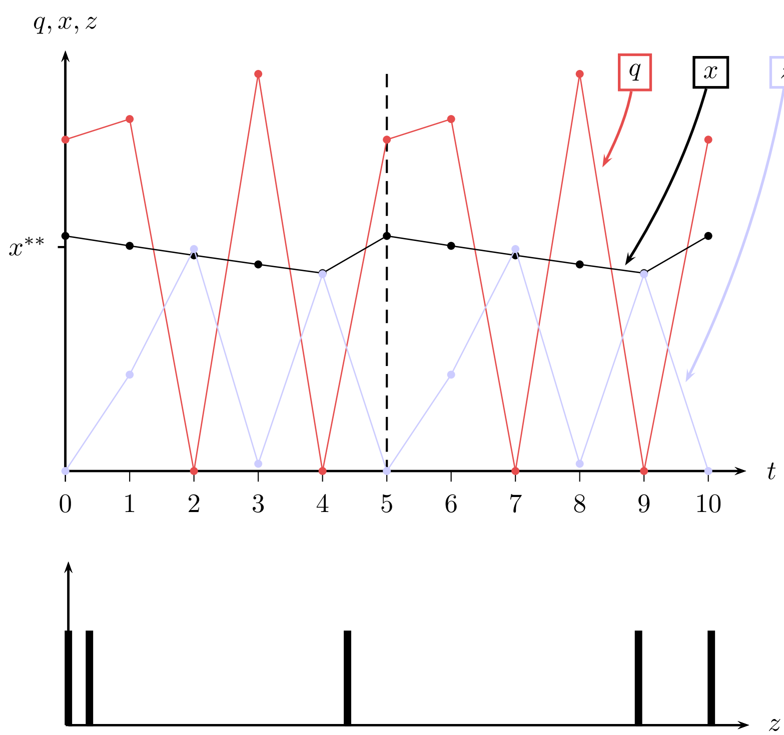

Figure 4: Equilibrium money cycle and related histogram of money holdings when $u(x) = x^{0.3}$, $c(q) = 0.1 + q^{1.5}$, $\beta = 0.98$ and $\pi = 0.01$.

Conjecture 4: Symmetric stationary equilibria always exist

In Theorem 3, we established that every symmetric stationary equilibrium is a money cycle equilibrium. But is there always such an equilibrium? We provide a counter-example. In the counter-example, there is no symmetric stationary equilibrium, but there is an asymmetric stationary equilibrium in which some agents follow one money cycle, and the other agents follow a different money cycle.

The counter-example is based on the setting in Figure 4. The preferences and production technology are the same, but the inflation rate is now fixed at $\pi = 0.01445$. To establish that there is no symmetric stationary equilibrium, we plot the symmetric excess demand correspondence, and show that it jumps over the market clearing point (0). In our model, the only endogenous price is the real value T of the (nominal) monetary transfers that are used to implement the inflation rate π. As the value of transfers increases, agents can afford to consume more and work less, in which case excess demand would increase. In the counter-example depicted in Figure 5, there is no market clearing value of T, so there is no symmetric stationary equilibrium. However, there are asymmetric equilibria in this setting.

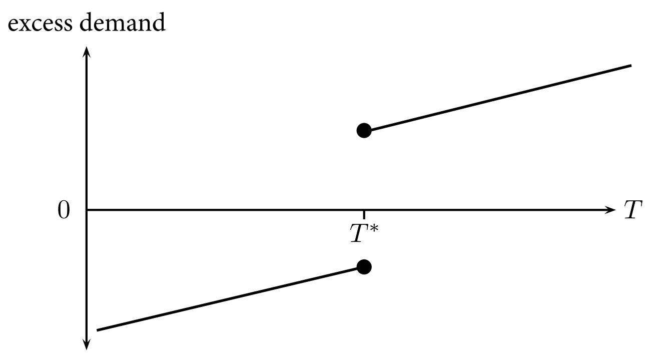

Figure 5: The symmetric excess demand correspondence jumps over the market-clearing level when $u(x) = x^{0.3}$, $c(q) = 0.1 + q^{1.5}$, $\beta = 0.98$, and $\pi = 0.01445$.

At the jump point T^*, the agents are indifferent between two different money cycle patterns – one of which involves excess demand and the other which involves excess supply. The excess demand and excess supply are of approximately equal magnitudes. If slightly more than half of the population followed one money cycle, and the rest followed the other, then markets would clear, forming an asymmetric equilibrium.[7]

[7] We conjecture that there always exists a stationary equilibrium with at most two different money cycles being followed by all agents.

Menzio *et al.* (2013) establish that a symmetric stationary equilibrium exists in their economy when there is no inflation or deflation. In their economy, the policy correspondences are single-valued and continuous, so jumps in the excess demand correspondence of the kind of Figure 5 do not arise. If the agents did not have access to perfectly fair lotteries, then we expect that only asymmetric stationary equilibria would exist for some parameter values.

7 Conclusion

This paper presents one of the simplest models of money possible – there are no search frictions, no overlapping generations, no cash-in-advance constraints, no nominal rigidities – just a fixed cost and a credit constraint. One might think it is hopeless to learn anything from such a trivial model. But we drew several striking conclusions: inflation has a "hot potato effect" leading to front-loading of consumption and under-production on work days; inflation can stimulate inefficiently high economic activity, even in the absence of shocks or asymmetric information; there is a non-degenerate distribution of money holdings; but wealthier agents do not necessarily consume more and work less, even when consumption and leisure are normal goods.

Our goal with this model was not to provide a theory of monetary stimulus or labour supply, but to understand better the key economic logic of money. Our model contributes to understanding monetary institutions in two ways. First, the frictions in more complex models are related to the simpler frictions in our model, and the equilibrium responses to the complex frictions mirror those in our simpler cousin. For example, the saving patterns in Kaplan and Violante (2014) mirror those in Baumol (1952) and Tobin (1956), and our model suggests how these might change if there were an increasing marginal cost of portfolio adjustment. Another example is in understanding the role of lotteries in Menzio *et al.* (2013), which were introduced as a technical device to obtain monotone policy functions. Our model suggests how their equilibria would change if agents faced overheads in playing lotteries.

Second, if a surprising conclusion of a simple model is quantitatively implausible, this suggests a course for future research: does the simple friction have a more realistic cousin that has a similar effect? For example, is there an overhead in some other economic activity (for example, financial portfolio adjustment) that leads inflation to have a stimulus effect?

A Liquid Lucas Trees

For purposes of comparison, we adapt the model from Section 2 by replacing money with deterministic Lucas trees that are in fixed supply. We show that the analysis above goes through with minor changes.

Suppose that instead of money, there is a forest of measure A of Lucas trees, which has an aggregate yield of yA every period. The price of the consumption good in time is p_t, and a measure one forest trades for R_t units of consumption good in time t. An agent with measure a of trees in time t has a value of

$$\begin{aligned} W_t(a_t) = &\max_{q_t \in \mathbb{R}_+, x_t \in \mathbb{R}_+, a_{t+1} \in \mathbb{R}_+} u(x_t) - c(q_t) + \beta W(a_{t+1}) \\ &\text{s.t.} \quad p_t x_t + p_t R_t a_{t+1} = p_t q_t + p_t (R_t + y) a_t. \end{aligned} \tag{13}$$

We focus our attention on equilibria in which $R_t = R$ is a constant. This leads to the stationary Bellman equation,

$$\begin{aligned} W(a) = &\max_{q \in \mathbb{R}_+, x \in \mathbb{R}_+, a' \in \mathbb{R}_+} u(x) - c(q) + \beta W(a') \\ &\text{s.t.} \quad x + Ra' = q + (R + y)a. \end{aligned} \tag{14}$$

This Bellman equation can be reformulated to look like (2), by making the state variable $\tilde{a} = (R + y)a$ the value (measured in consumption units) of the forest a:

$$\begin{aligned} V(\tilde{a}) = &\max_{q \in \mathbb{R}_+, x \in \mathbb{R}_+, \tilde{a}' \in \mathbb{R}_+} u(x) - c(q) + \beta V(\tilde{a}') \\ &\text{s.t.} \quad x + \frac{R}{R + y} \tilde{a}' = q + \tilde{a}. \end{aligned} \tag{15}$$

Unless the trees have no yield (i.e. $y = 0$), this is *not* an isomorphic representation, because there is no equivalent transfer term T. Nevertheless, it is clear that the theory from Section 5 generalises in a straight-forward way, as the transfers do not play an important role in the proofs.

On the other hand, this reformulation is confusing for thinking about market clearing conditions, so we define equilibrium in terms of the original formulation. Let F be the measure of agents holding up to a units of forest. The stationary symmetric equilibrium market clearing conditions are:

$$\int a'(a) dF(a) = A$$

$$\int x(a) dF(a) = \int q(a) dF(a) + Ay.$$

It is straightforward to show that the utilitarian social planner still prefers each agent to consume x^{**} and work either $q \in \{0, \bar{q}\}$, as before. In other words, since there is an exogenous endowment available in the economy, a smaller measure of agents need to work to sustain the same amount of consumption.

If the trees have no yield ($y = 0$), then the economy is equivalent to a monetary economy with no inflation or deflation ($\pi = 0$). If trees have a positive or negative yield ($y \neq 0$), then the Lucas trees function like money with growth rate $\pi = \frac{R}{R+y} - 1$. In other words, positive yields are like deflation and negative yields are like inflation.

We observe that the only property of the forest that matters is the aggregate yield. That is, the equilibria do not change when the size of the forest doubles and the yield of each tree halves, because agents can bundle two unproductive trees into a single productive tree. Equilibria can only be efficient if the aggregate yield is a sufficiently large "subsidy" to support a return equal to the rate of time-preference.

B Algorithm

The usual approach to calculating dynamic equilibria is to exploit the contraction property of Bellman operators to approximate the agent's value function. However, the discrete choices make the value function non-concave, and therefore difficult to represent faithfully on a computer. Our approach is to try every possible sequence of extensive margin choices, and to apply Theorem 4 to calculate the optimal choices on the intensive margins, taking the extensive margins as fixed. The main difficulty is that, a priori, there might be an (uncountably) infinite number of possible extensive margin choices among all possible money cycles. We solve this problem with Theorem 5, which establishes an upper bound on the length of a money cycle equilibrium in a given economy, and hence a bound on the number of possible extensive margin choices the agent has.

Theorem 5. *Consider a money cycle equilibrium with inflation so that $T > 0$.*

(i) If the agent holds real balances of z_1, then

(a) their consumption x_1 lies in $[T, \bar{x}_1]$, where $\bar{x}_1 = \max\{z_1 + T, \hat{x}\}$ and $\hat{x}$ solves $u'(x) = c'(x - z_1 - T)$.

(b) they spend all of their money within the following number of periods,

$$\left\lceil \log \frac{u'(\bar{x}_1)}{u'(T)} \Big/ \log \frac{\beta}{1+\pi} \right\rceil .$$

(ii) Since money cycles begin with $z_1 = 0$, the length of money cycles is bounded by this expression at $z_1 = 0$.

Proof. (i) (a) Clearly $x_1 \geq T$. We need to show that $x_1 \leq \max\{z_1 + T, \hat{x}\}$. If the agent does not work in the first period, then $x_1 \leq z_1 + T$. If the agent works, we will show that $x_1 \leq \hat{x}$. Intuitively, if an agent consumes a lot, then they must also produce a lot; but as diminishing marginal utility and increasing marginal cost set in, it becomes suboptimal to increase consumption and production. Since the agent starts with z_1 real balances of money, the budget constraint implies that $q_1 \geq x_1 - z_1 - T$. The first-order conditions imply $u'(x_1) = c'(q_1)$. Moreover, since marginal cost is increasing, $c'(q_1) \geq c'(x_1 - z_1 - T)$. Thus, $u'(x_1) \geq c'(x_1 - z_1 - T)$, or equivalently, $x_1 \leq \hat{x}$ since u' is decreasing and c' is increasing.

(b) Now suppose that $z_2, \ldots, z_n > 0$. We will put an upper bound on n for which this can be true. Under inflation, $x_n \geq T > 0$. By the Euler equation,

$$u'(x_1) = \left(\frac{\beta}{1+\pi}\right)^n u'(x_n).$$

Substituting the bound for x_1 above and the bound $x_n \geq T$ into this equation gives

$$u'[\max\{z_1 + T, \bar{x}_1\}] \leq u'(x_1) = \left(\frac{\beta}{1+\pi}\right)^n u'(x_n) \leq \left(\frac{\beta}{1+\pi}\right)^n u'(T),$$

which can be rearranged to the bound on n given above.

(ii) Trivial.

□

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