Edinburgh School of Economics

Discussion Paper Series

Number 227

Unemployment Risk and Wage Differentials

Gene M Grossman
(Princeton University)

Elhanan Helpman
(Harvard University and CIFAR)

Philipp Kircher
(The University of Edinburgh)

Date
September 2013

Published by
School of Economics
University of Edinburgh
30 -31 Buccleuch Place
Edinburgh EH8 9JT
+44 (0)131 650 8361
http://edin.ac/16ja6A6

THE UNIVERSITY *of* EDINBURGH

4.4 The Effects of Trade on Wage and Salary Distributions

We turn now to the effects of trade on wages and salaries. In Section 3, where we studied an economy with homogeneous managers, we identified two considerations that color the link between output prices and factor prices. First, when trade causes the relative price of a country's export good to rise, the expansion of the export sector tends to benefit all types of the factor used intensively in that sector and to harm the factor used intensively in the import-competing sector. Second, when a factor is heterogeneous, trade tends to benefits those types of the factor that have comparative advantage in the export industry and to harm those types that have comparative advantage in the import-competing industry. Of course, these two influences on factor prices are familiar from the Heckscher-Ohlin economy and the Ricardo-Viner economy, respectively.

In an economy with two heterogeneous factors and complementarities between their abilities (or qualities), a new consideration comes into play. When productivity in each sector is a strictly log supermodular function of the employees' ability levels, the general equilibrium determines the matching of workers and managers within production units. Then, as output prices change and factors are re-allocated between sectors, the inflows of some marginal types into the expanding sector and the outflows of these types from the contracting sector causes a re-matching of types in each industry. This re-matching in turn affects each type's productivity and therefore the equilibrium rates of pay. We will find that re-matching introduces a mechanism by which trade alters within-industry wage and salary distributions.[28]

For concreteness, consider a country that exports good 2. In Figure 7, the solid curves *cd* and *ab* depict the country's (inverse) matching function prior to the opening of trade for the case of an HL/LH equilibrium in which the more able workers and less able managers sort to industry 1. To understand the distributional implications of trade in such a country, we examine the effects of an increase in the relative price of good 2. This draws workers and managers into sector 2, so that q_H^* falls and q_L^* rises.[29] The new boundary points are represented by c', d', a' and b'. As is evident from the figure, the new inverse matching function (represented by the broken curves) lies below the original function for all worker and manager types that remain in their original industry of employment. As a result, the opening of trade allows all managers except those that switch sectors to achieve better matches than before, while causing all workers except those that switch sectors to realize worse matches than before.

Proposition 8 summarizes these effects of trade on matching for the case of an HL/LH equi-

[28] We have also studied the effects of trade on factor prices in an economy with Cobb-Douglas productivity and report our findings in the appendix. As we have noted, the matching of managers and workers is not well determined in such an economy, since the relative productivity of two types of worker, for example, is not affected by the ability level of the manager with whom they might be matched. In such a setting, the rematching that results from trade is not determined, but neither is it material for factor prices. We find, with Cobb-Douglas productivity, that trade has no effect on within-industry wage or salary distribution, and the Stolper-Samuelson and Ricardo-Viner influences on factor prices are analagous to those in an economy with homogeneous managers.

[29] Before any factor reallocation, the increase in p_2 raises the value marginal product of the marginal workers and managers in sector 2 relative to those in sector 1. As factors reallocate, marginal products change and rematching occurs. But we show in the appendix that these secondary effects cannot overturn the impact effects, so that q_H^* must fall and q_L^* must rises in the setting described by the figure.

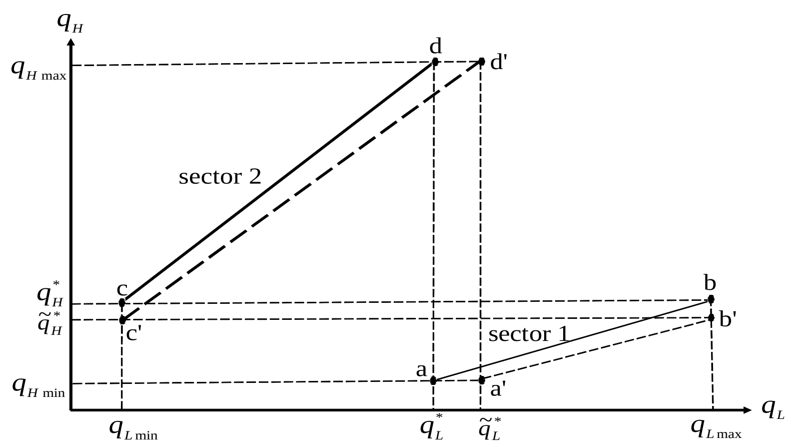

Figure 7: Effects of a rise in p_2 on matching: HL/LH equilibrium

librium and reports the implications for wage and salary inequality.

Proposition 8 *Suppose that: (i) Assumption 2 holds and (ii) the initial equilibrium is a threshold equilibrium with an HL/LH sorting pattern. Then an increase in p_2 (a) raises the labor cutoff q_L^* and reduces the manager cutoff q_H^* so that more workers and more managers are employed in sector 2; (b) worsens the matches for all workers except those that switch from sector 1 to sector 2; (c) improves the matches for all managers except those that switch from sector 1 to sector 2; (d) reduces within-industry wage inequality in both sectors and overall wage inequality in the economy; and (e) increases within-industry salary inequality in both sectors and overall salary inequality.*

Evidently, wage inequality falls among workers originally in industry 2 and among those remaining in industry 1. Take for example any two workers q_L^c and $q_L^{c'}$ such that $q_{L\min} \leq q_L^c < q_L^{c'} \leq q_L^*$. Both workers see their match deteriorate as a result of the increase in the price of good 2, but the re-matching harms the worker with ability $q_L^{c'}$ by relatively more due to the complementarities between factor types. This can be seen from (20), wherein the strict log supermodularity of $\psi(\cdot)$ implies that a downward shift in $\mu(\cdot)$ reduces the integrand on the right-hand side and thus reduces the relative wage of the more able worker in the pair. The same is true for any pair of workers with abilities between $\tilde{q}_L^*$ and $q_{L\max}$. Finally, consider a pair of workers that switch sectors; i.e., those that have ability levels between q_L^* and $\tilde{q}_L^*$. The relative wage of the less able worker in this pair must rise, because the elasticity of the wage schedule in (17) is determined after the price change by the elasticity ratio in sector 2, whereas before it was determined by the elasticity ratio in sector 1. Since the more able workers sort to sector 1, it must be that the former elasticity is smaller than the latter. It follows that wage inequality declines also among workers that switch sectors and therefore among all workers in the economy; see Figure 8 for an example.

What is the overall effect of the price change on the welfare of the various workers? There are

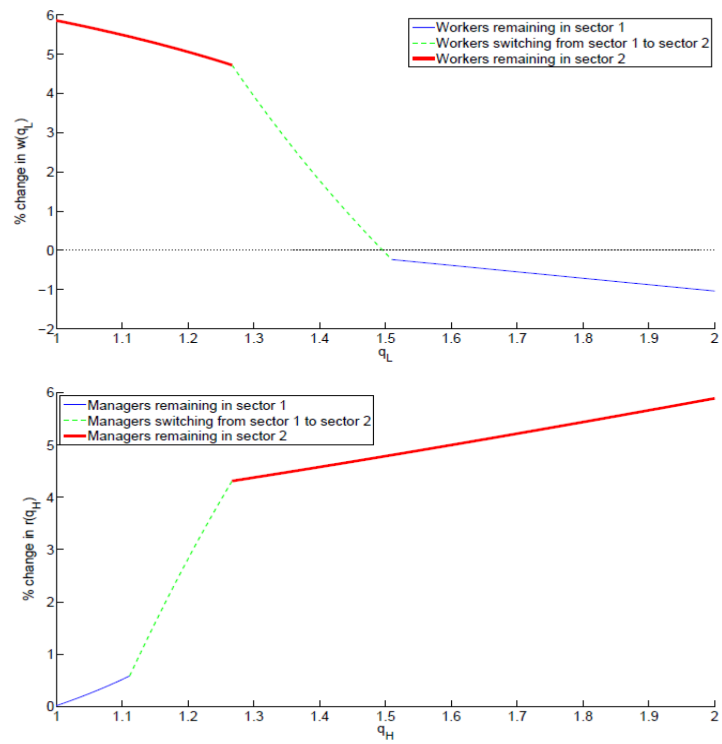

Figure 8: Effects of a 5% increase in p_2 on wages and salaries in an HL/LH equilibrium

several possibilities that can emerge, as can be seen in the numerical simulations presented by Lim (2013). First, if sector 1 is labor intensive and the difference in factor intensities across sectors is large relative to the specificity of the heterogeneous factors, then the Stolper-Samuelson forces dominate. In such circumstances, real wages decline for all workers while real salaries increase for all managers. Of course, if sector 2 is the labor-intensive industry, then the opposite outcomes are possible, with real gains for all workers and losses for all managers.

Figure 8 depicts the wage and salary responses for a less extreme case.[30] Here, sector 2 is labor intensive and p_2 rises by 5%. All workers initially in sector 2 see their wages rise and those at the bottom end of the ability distribution enjoy a wage hike in excess of 5%. Meanwhile, the workers who remain in sector 1 suffer a decline in wages despite the rise in the price of the labor-intensive good. These workers suffer from their comparative disadvantage in the expanding sector. As for managers, those at the top end of the ability distribution gain the most and some see salary improvements in excess of 5%. Those at the bottom of the ability distribution enjoy welfare gains only if they devote little of their income to the export good. The figure shows the widening of salary inequality among managers.

A host of other possible configurations can emerge, but all can be understood similarly with reference to the relevant factor intensities and sector specificities; see Lim (2013) for examples. Rather than dwell on these cases, we turn now to the wage and salary effects of trade in an

[30] See Lim (2013) for the parameter values and functional forms that underlie this figure.

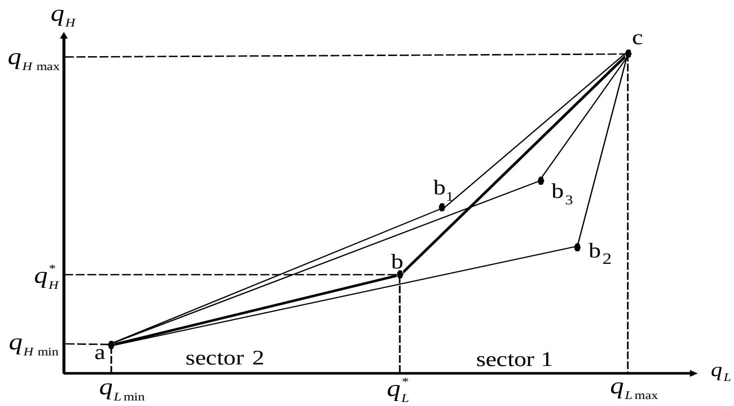

Figure 9: Impact of a rise in p_2 on matching: HH/LL equilibrium

HH/LL equilibrium. Recall the matching and sorting patterns for such an equilibrium that were displayed in Figure 6. We show in the appendix that, when the price of good 2 rises in such a setting, sector 2 expands by attracting both additional workers and additional managers. It follows that both q_L^* and q_H^* rise. In this case, the implications for matching vary according to whether the movement of workers or the movement of managers dominates.

Figure 9 illustrates the various possibilities.[31] The thick curve abc represents the initial inverse matching function. Now suppose that q_L^* rises only modestly, while q_H^* rises more dramatically.[32] Then the new equilibrium would be represented by an inverse matching function such as ab_1c. In the event, all workers' matches improve following the price hike, whereas all managers see their matches deteriorate. Alternatively, the inflow of workers to sector 2 can be large relative to that for managers, in which case q_L^* could expand greatly compared to the expansion in q_H^*. This possibility is illustrated by the inverse matching function ab_2c in the figure, and it implies a deterioration in match quality for all workers and an improvement for all managers. Finally, the inverse matching function ab_3c depicts an intermediate case. Notice that the matches improve for all workers initially in sector 2 but deteriorate for all those remaining in sector 1.

Let us focus on the case where the outcome is an inverse matching function such as ab_1c to discuss the implied wage and salary responses. Since workers' matches improve, wages rise faster with ability than before. Since managers' matches deteriorate, the opposite is true of managerial salaries. Notice that the inverse matching function has a steeper slope at point a in the new equilibrium than before the price change. It follows from Lemma 6 that the wage of the least able workers must rise. These workers benefit directly from the increase in p_2 and indirectly from the improvement in their matches. The direct benefit alone matches the proportional increase in price, so these workers enjoy real income gains. At the opposite end of the spectrum, the most able

[31] Lim (2013) provides numerical examples of each along with the underlying parameter values.

[32] This outcome plausibly arises when sector 2 is considerably more manager intensive than sector 1.

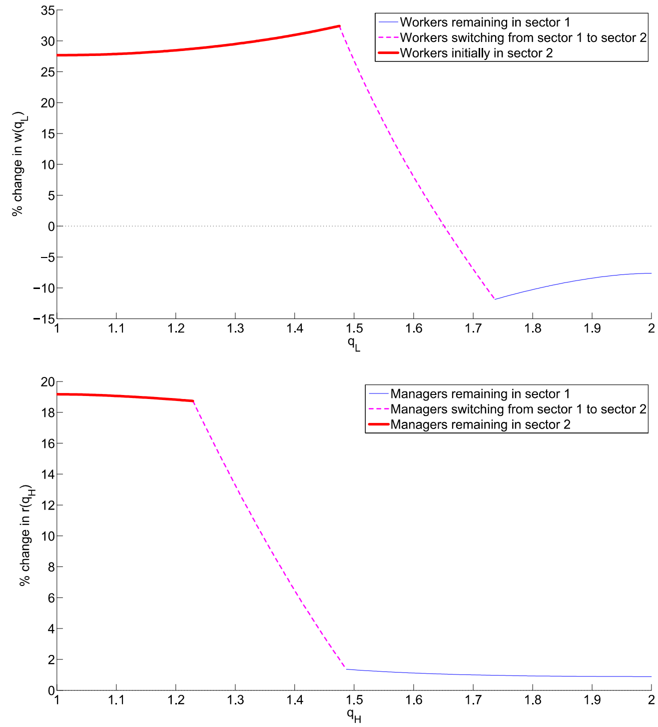

Figure 10: Effects of a 20% increase in p_2 on wages and salaries in an HH/LL equilibrium

workers must lose. The change in p_2 has no direct effect on their value marginal product. Since the new inverse matching function is flatter at point c than the initial function, Lemma 6 implies that these workers suffer a decline in nominal wages. The gain in real income for the least able workers and the loss for the most able workers represents a narrowing of wage inequality across sectors, whereas the improved matching implies that wages are more unequal within each sector.

Figure 10 presents another example drawn from Lim (2013). Notice that the least able workers enjoy real income gains, though not as large as for those more able than themselves who initially are employed in the same sector. Meanwhile, the most able workers lose, but not as much as those less able than themselves who remain in sector 1. The figure also shows the effect on managerial salaries. In this example, all managers realize income gains in terms of good 2 but losses in terms of good 1. These gains are smaller and the losses larger as we move up the salary distribution. A decline in $r\left(q_{H\min}\right)/p_2$ is guaranteed in this case, because the direct effect for the least able managers is a salary increase proportional to the rise in p_2, but the steepening of the inverse matching function at a implies that their salaries must fall relative to the price of what they produce. The rise in $r\left(q_{H\max}\right)/p_1$ also is guaranteed, because the inverse matching function is flatter at point c than before. Finally, we know that the new salary function is flatter than the old both for managers initially in sector 2 and for those that remain in sector 1, because the deterioration in match quality hits especially hard for the more able managers in any sector.

If the inverse matching function instead is qualitatively like that depicted by ab_2c in Figure 9, then the outcomes are just the opposite. Low-ability managers gain from an increase in p_2, because their value marginal product rises in proportion to the price hike and rises further as a result of the re-matching. High-ability managers lose in real terms, because $r(q_{H\max})/p_1$ falls. All wages rise, albeit less than in proportion to the price increase. The wage hikes are proportionally greatest for those at the bottom end of the ability distribution. As a result of these factor price responses, wage inequality declines both within and between sectors, whereas salaries become more unequal within sectors, but those at the bottom who are employed in sector 2 gain relative to those at the top who are employed in sector 1.

Finally, if the inverse matching function is like that depicted by ab_3c, then the outcomes are a mix of those described above. In this case, all workers initially employed in sector 2 must benefit from the price increase, while all managers initially employed in sector 1 must lose. The low-ability managers and the high-ability workers both gain in compensation relative to the price of good 1, but lose relative to the price of good 2. Lim (2013) provides numerical examples.

5 Labor Market Frictions

Until now, we have assumed that labor markets flawlessly and costlessly allocate the various types of labor to their most efficient uses. Of course, the smooth functioning of labor markets is notoriously suspect and worker heterogeneity would only seem to exacerbate the potential difficulties. In this section, we show how a simple form of search frictions can be incorporated into the analysis. The extension allows us to discuss the distribution of unemployment rates across the ability spectrum alongside the distribution of wages.

To keep matters simple, we continue to assume a frictionless market for managers. In other words, firms can hire managers of whatever ability and in whatever numbers they wish by offering a competitive salary.[33] But firms must search for their workers and workers for jobs. We follow Peters (1991, 2000), Acemoglu and Shimer (1999), Burdett et al. (2001), Eeckhout and Kircher (2010a, 2012), and others in modeling labor-market frictions with "directed search," whereby firms post costly "vacancies" that announce their compensation offers and targeted workers and employee-seeking firms meet randomly. In particular, we extend the approach of Eeckhout and Kircher that allows for worker heterogeneity and multiple hires per firm to an environment with two industries.

Suppose, as before, that the output in industry i of a production unit comprising a manager of ability q_H and ℓ workers of ability q_L is given by (1). A firm (or entrepreneurial manager) hires workers by posting vacancies. Each posting costs c_i units of the the firm's final output. A posting lists the ability level q_L that the firm targets and the wage ω that it will pay to any employee of this type. We assume that the firm can commit to these job attributes, in the sense that it will not hire workers with ability different from the posted level nor attempt to renegotiate its wage offer

[33] Perhaps the best way to justify this assumption is to imagine the manager as an entrepreneur, as in Lucas (1978). Then it is the manager that searches for employees and her salary amounts to the residual profits after wages and hiring costs are paid. Alternatively, one might think of the second factor as being *capital*, instead of managers, in which case an assumption that firms can readily find machines of the quality they desire is not so hard to swallow.

after it meets with a job applicant.[34] The firm chooses v, the number of its vacancies, to maximize profits.

Workers are risk neutral. Each worker applies for a single job of his choosing.[35] Workers consider only the jobs for which they are qualified, because firms will not hire types different from those targeted in their announcements. Among relevant jobs, each worker applies for the position that offers the greatest expected income. In equilibrium, workers must be indifferent among the range of openings posted for their type.

Let s be the number of workers seeking jobs at a firm that has posted v vacancies. We assume that the search process results in the consummation of $M(s,v)$ jobs, where

$$M(s,v) = Bs^{\tau}v^{1-\tau}, \tag{21}$$

$B > 0$ and $0 < \tau < 1$.[36] For a firm, the probability of filling any given vacancy is $\delta_v(s/v) = B(s/v)^{\tau}$, whereas for a worker the probability of a successful application is $\delta_s(s/v) = B(s/v)^{-(1-\tau)}$. The former is increasing in s/v, while the latter is decreasing in s/v; i.e., a firm's chances of filling a vacancy improve and a worker's chances of landing a job decline with the number of applicants per posting.

Now let $w(q_L)$ be the *expected wage* that workers of type q_L obtain in equilibrium, which each firm takes as given. A firm must offer at least this expected wage or it will find itself without applicants; and it has no reason to offer more. In equilibrium, a firm with v vacancies that offers a wage ω targeted to workers with ability q_L attracts s applicants, where s is such as to make the applicants indifferent between the firm's openings and their other opportunities; i.e., s solves $\delta_s(s/v)\omega = w(q_L)$. Using (21), this can be rewritten as

$$\frac{s}{v} = \left[\frac{B\omega}{w(q_L)}\right]^{\frac{1}{1-\tau}}. \tag{22}$$

Equation (22) is the main building block in a model with directed search; it ties the wage announcement ω to the endogenous number of applications per vacancy s/v, which in turn determines the firm's fill rate, $\delta_v(s/v)$.[37] Given the expected wage $w(q_L)$, the firm can use (22) to compute

[34] Alternatively, we could allow a firm to post a wage schedule and to hire any worker it happens to meet at the wage specified by the schedule. If each vacancy generates at most one meeting with a job applicant, then it is never optimal for the firm to induce applications from more than one type of worker; see Eeckhout and Kircher (2010a, 2010b) for proof of this assertion in related environments. In such circumstances, there is no loss of generality in assuming that the firm targets only one type of worker. Shimer (2005) studies a setting in which one vacancy can result in multiple meetings with potential employees. Then, in the general, it is optimal for any firm to induce applications from several different types. We do not explore this possibility here.

[35] This assumption is common in the literature on direct search. Galenianos and Kircher (2009) describe settings in which the restriction to one application per worker does not change the qualitative predictions of the model.

[36] The job-search literature refers to $M(s,v)$ as a "matching function" but we eschew that terminology so as to avoid confusion with the function that "matches" workers and managers, $q_L = m(q_H)$. The Cobb-Douglas form for $M(\cdot)$ is common in the literature, and is implicitly coupled with the usual restriction that B is sufficiently small to imply meeting probabilities below unity for both vacancies and workers.

[37] Peters (1991, 2000) and Burdett et al. (2001) provide microfoundations for a relationship similar to (22). They begin by assuming a finite number of jobs and vacancies and then allow the economy to grow large without bound.

the number of workers that will seek its employment and thus the number of workers $\ell = M(s, v)$ that it will succeed in hiring. Again using (21), together with (22), we see that a firm that posts v vacancies targeted at workers with ability q_L and that offers a wage of ω manages to hire ℓ workers, where

$$\ell = B^{\frac{1}{1-\tau}} \left[\frac{\omega}{w(q_L)} \right]^{\frac{\tau}{1-\tau}} v. \tag{23}$$

Evidently, hires are proportional to the number of vacancies and rise with the ratio of the firm's wage offer to the workers' outside option.

Now consider the profit-maximization problem facing a firm with a manager of ability q_H that chooses to operate in industry i. The firm pays $p_i c_i v$ to post v vacancies and pays ω to each of the ℓ workers that it eventually hires. Its profits are given by

$$\pi_i = p_i \psi_i(q_H, q_L) \ell^{\gamma_i} - \omega \ell - p_i c_i v - r(q_H),$$

where $r(q_H)$ as before represents the manager's salary. Then, using (22) and the first-order condition for the firm's optimal choice of wage offer, we can re-express its profits as

$$\pi_i = p_i \varphi_i(q_H, q_L) s^{\zeta_i} - w(q_L) s - r(q_H),$$

where

$$\varphi_i(q_H, q_L) \equiv [1 - (1-\tau)\gamma_i] \left[\frac{(1-\tau)\gamma_i}{c_i} \right]^{\frac{(1-\tau)\gamma_i}{1-(1-\tau)\gamma_i}} B^{\frac{\gamma_i}{1-(1-\tau)\gamma_i}} \psi_i(q_H, q_L)^{\frac{1}{1-(1-\tau)\gamma_i}}$$

and

$$0 < \zeta_i \equiv \frac{\tau \gamma_i}{1 - (1-\tau)\gamma_i} < 1.$$

Notice that this expression for profits has the same mathematical properties as the profit function $\pi_i = p_i \psi_i(q_H, q_L) \ell^{\gamma_i} - w(q_L)\ell - r(q_H)$ that we encountered in Section 4, because if $\psi_i(q_H, q_L)$ satisfies part (ii) of Assumption 3 (i.e., it is strictly increasing, continuously differentiable, and strictly log supermodular) so too does $\varphi_i(q_H, q_L)$, and ζ_i like γ_i is between zero and one.[38] In other words, the firm's choice about the number of job applications to invite in a setting with search frictions is much like its choice about the number of workers to hire in a setting without them. The first-order condition for s implies

$$s = \left[\frac{\delta_i p_i \varphi_i(q_H, q_L)}{w(q_L)} \right]^{\frac{1}{1-\delta_i}}, \tag{24}$$

This generates a balls–and-urns type function for applicants per vacancy, rather than the Cobb-Douglas form that is more commonly assumed. Galenianos and Kircher (2012) extends their setup to generate CES and Cobb-Douglas matching functions. With but a few exceptions, the literature on directed search specifies the matching function individually for each vacancy, and we follow in this tradition.

[38] Note too that if $\psi_i(q_H, q_L)$ is a product of power functions, so too is $\varphi_i(q_H, q_L)$. And if $\psi_i(q_H, q_L)$ has a constant elasticity of substitution between q_H and q_L, so too does $\varphi_i(q_H, q_L)$.

which generates the profit function

$$\pi_i(q_H, q_L) = \bar{\zeta}_i p_i^{\frac{1}{1-\delta_i}} \varphi_i(q_H, q_L)^{\frac{1}{1-\zeta_i}} w(q_L)^{-\frac{\zeta_i}{1-\zeta_i}} - r(q_H),$$

where $\bar{\zeta}_i \equiv \zeta_i^{\frac{\zeta_i}{1-\zeta_i}}(1-\zeta_i)$. This expression has much the same form as (14), which applies in the absence of search frictions. Finally, the analog to the labor-market clearing condition from before is the requirement that the aggregate number of applications induced by firms operating in industry i and targeting workers of ability q_L must equal the number of workers with that ability level that sort to the sector in search of a job. With these observations, we conclude that the equilibrium expected wage function $w(q_L)$, salary function $r(q_H)$ and matching function $q_L = m(q_H)$ can be characterized as the solution to three differential equations analogous to (17)-(19), a zero profit condition analogous to (16), and a set of boundary conditions. Evidently, comparative advantage again derives from a country's relative factor endowments and its distributions of worker and manager ability. Moreover, since $\zeta_1 > \zeta_2$ if and only if $\gamma_1 > \gamma_2$, the cross-sectoral differences in factor intensities interact with differences in factor endowments to determine the pattern of trade in much the same way as before. The search frictions themselves are not an independent source of comparative advantage so long as these frictions are similar in the two sectors.[39]

The model with search frictions features different employment rates across the range of ability levels. In order to discuss the impact of trade on employment, we combine the optimal choice of wage offer with a firm's desired number of applications per manager to derive

$$\omega(q_L) = B^{-\gamma_i} \left(\frac{1-\tau}{\tau p_i c_i}\right)^{-(1-\tau)\gamma_i} w(q_L)^{1-(1-\tau)\gamma_i}.$$

The expected wage $w(q_L)$ must be an increasing function of ability. It follows that, among workers that seek employment in a given industry i, those with greater ability see higher posted wages for the jobs they pursue. Next, we substitute this expression for $\omega(q_L)$ into (23) to derive an expression for the employment rate for workers of ability q_L, namely

$$\frac{\ell}{s} = B^{\gamma_i} \left(\frac{1-\tau}{\tau c_i}\right)^{(1-\tau)\gamma_i} \left[\frac{w(q_L)}{p_i}\right]^{(1-\tau)\gamma_i}. \tag{25}$$

Since the expected wage on the right-hand side is an increasing function of ability, we conclude that so too is the employment rate among workers seeking jobs in a given industry. We record our findings in

Proposition 9 *Suppose that Assumption 2 holds. Let $q_L', q_L'' \in Q_i$, with $q_L' > q_L''$. Then the job listings targeted to workers with ability q_L' offer a higher expected wage and a greater probability of employment than those targeted to q_L''. The opening of trade causes the within-sector inequality of expected wages and employment rates to move in the same direction.*

[39] If the number of meetings in (21) varies by sector, then it is immediate from the definition $\zeta_i \equiv \tau_i\gamma_i/(1-(1-\tau_i)\gamma_i)$ that the search process constitutes an additional source of comparative advantage.

In a setting with search frictions, the opening of trade affects differently the employment rates at different ability levels. Let us consider just one example to illustrate how the analysis can be performed. Suppose a country has an HL/LH sorting pattern such as that depicted in Figure 5 and that the country exports good 2. The opening of trade generates an increase in p_2. Figure 7 shows the effects of such a price change on the matching of worker and manager types in each sector. As we have seen, the workers who do not switch sectors find themselves teamed with a less able manager than before. Now, Figure 8 can be interpreted as illustrating the predicted impact on *expected* wages. The figure shows an increase in $w(q_L)/p_2$ for some of the least able workers, who sort to sector 2, a decline in $w(q_L)/p_2$ for some moderately able workers that sort to sector 2, and a decline in $w(q_L)/p_1$ for the most able workers, who sort to sector 1.

We refer now to equation (25), which applies in the presence of search frictions. The equation implies that the employment rate rises for the aforementioned group of least able workers while it falls for those with moderate and high ability. Overall, the distribution of employment rates becomes more equal across the worker population. Of course, the effects of trade on the distribution of employment would be just the opposite if the country instead imported good 2. Evidently, trade can widen or narrow the inequality in employment rates across the ability distribution according to the sorting pattern that is realized and the comparative advantage of the country. The determinants of these outcomes in an economy with directed search are similar to the determinants of wage inequality in an economy that has frictionless labor markets.

6 Concluding Remarks

In this paper, we have extended the familiar two-sector, two-factor model of international trade to include heterogeneous factors of production. In a model with factor heterogeneity, we can examine the determinants of factor sorting to industries and the determinants of factor matching within industries. When the productivity of a production unit depends on both the manager's and workers' abilities—and particularly when there are strong complementarities between the two—the forces that guide sorting and matching become inextricably linked. The economy-wide pattern of factor assignments can be subtle and complex even in the presence of strong complementarities that dictate positive assortative matching within every sector.

A model with heterogeneous factors allows a more complete analysis of the distributional effects of trade than is possible in one with homogeneous factors. In particular, we can ask how the opening of trade or trade liberalization affects the wage and salary distributions over the entire range of compensation levels. In general, there are three considerations that determine the effects of trade on the income of a particular individual. First, as in the standard Heckscher-Ohlin world with homogeneous factors, there is the question of whether the export sector is intensive in the use of workers or managers. Second, as in the standard Ricardo-Viner world with factor specificity, there is the question of whether an individual's type generates a personal comparative advantage in the export sector or the import-competing sector. Finally, and most novel, there is the question of how trade affects the individual's match with other factors of production. If a change in trade conditions causes a worker to re-match with a better manager than before, then his productivity will improve and his wage will receive an upward boost. If instead a worker's match deteriorates, then his wage may suffer. Interestingly, the effects of trade on wage or salary inequality across sectors may run counter to the effects on inequality within a sector.

We have shown that the Heckscher-Ohlin theorem extends to a setting with heterogeneous factors provided that the countries share similar distributions of worker and managerial talent. But we have also noted how differences in the distributions of talent can be an independent source of comparative advantage. A country that has more able workers than another—in the sense of a rightward shift in the talent distribution—will produce relatively more of the good for which productivity responds more elastically to ability.

Finally, we have incorporated search frictions. In a simple setting with directed search, firms create vacancies and make wage offers to workers of a targeted type. In such a setting, trade affects not only the distribution of wages but also the distribution of employment rates across the different types of workers. We provide an example in which the main insights from the earlier analysis carry over without modification to an environment with unemployment. But much work remains to the elucidate the connection between trade and the efficiency of matching and to understand how globalization affects equilibrium unemployment rates for different types of workers.

References

[1] Acemoglu, Daron and Shimer, Robert, 1999. "Holdups and Efficiency With Search Frictions," *International Economic Review 40*(4), 827–849. "

[2] Antràs, Pol, Garicano, Luis and Rossi-Hansberg, Esteban, 2006. "Offshoring in a Knowledge Economy," *Quarterly Journal of Economics 121(1)*, 31-77.

[3] Bombardini, Matilde, Gallipoli, Giovanni and Pupato, Germán, 2012. "Skill Dispersion and Trade Flows," *American Economic Review 102(5)*, 2327-48.

[4] Burdett, Kenneth, Shi, Shouyong. and Wright, Randall, 2001."Pricing and Matching With Frictions," *Journal of Political Economy 109*(5), 1060–1085.

[5] Costinot, Arnaud, 2009. "An Elementary Theory of Comparative Advantage," *Econometrica 77(4)*, 1165-92.

[6] Costinot, Arnaud and Vogel, Jonathan, 2010. "Matching and Inequality in the World Economy," *Journal of Political Economy 118(4)*, 747-86.

[7] Eeckhout, Jan and Kircher, Philipp, 2010a. "Sorting and Decentralized Price Competition," *Econometrica 78*(2), 439-74.

[8] Eeckhout, Jan and Kircher, Philipp, 2010b. "Sorting vs. Screening - Search Frictions and Competing Mechanisms," *Journal of Economic Theory 145*(4), 1354-85.

[9] Eeckhout, Jan and Kircher, Philipp, 2012. "Assortative Matching with Large Firms: Span of Control over More versus Better Workers." Downloadable at http://personal.lse.ac.uk/kircher/Papers/sorting-and-factor-intensity.pdf

[10] Galenianos, Manolis and Kircher, Philipp, 2009."Directed Search with Multiple Job Applications," *Journal of Economic Theory 114*(2), 445-71.

[11] Galenianos, Manolis and Kircher, Philipp, 2012. "On the Game-theoretic Foundations of Competitive Search Equilibrium," *International Economic Review 53*(1), 1-21.

[12] Garicano, Luis, 2000. "Hierarchies and the Organization of Knowledge in Production," *Journal of Political Economy 108(5)*, 874-904.

[13] Grossman, Gene M., 1983. "Partially-Mobile Capital: A General Approach to Two-Sector Trade Theory," *Journal of International Economics 15(1-2)*, 1-17.

[14] Grossman, Gene M. and Maggi, Giovanni, 2000. "Diversity and Trade," *American Economic Review 90(5)*, 1255-75.

[15] Helpman, Elhanan, Itskhoki, Oleg, Muendler, Marc-Andreas and Redding, Stephen, 2012. "Trade and Inequality: From Theory to Estimation," NBER Working Paper No. 17991.

[16] Jones, Ronald, 1971. "A Three-Factor Model in Theory, Trade and History," J. Bhagwati, R. Jones, R. Mundell and J. Vanek, eds., *Trade, Balance of Payments and Growth*, Amsterdam: North-Holland.

[17] Lim, Kevin, 2013. "Numerical Simulations for 'Matching and Sorting in the Global Economy'." Downloadable at http://www.princeton.edu/~grossman/grossman_working_papers.htm.

[18] Lucas, Robert E., Jr., 1978. "On the Size Distribution of Business Firms," *The Bell Journal of Economics 9*(2), 508-23.

[19] Mussa, Michael, 1982. "Imperfect Factor Mobility and the Distribution of Income," *Journal of International Economics 12(1-2)*, 125-141.

[20] Ohnsorge, Franziska and Trefler, Daniel, 2007. "Sorting it Out: International Trade with Heterogeneous Workers," *Journal of Political Economy 115(5)*, 868-92.

[21] Peters, Michael, 1991. "Ex ante Price Offers in Matching Games: Non Steady States," *Econometrica 59*(5), 1425–54.

[22] Peters, Michael, 2000. "Limits of Exact Equilibria for Capacity Constrained Sellers With Costly Search," *Journal of Economic Theory 95*(2), 139–68.

[23] Ruffin, Roy, 1988. "The Missing Link: The Ricardian Approach to the Factor Endowment Theory of International Trade," *American Economic Review 78(4)*, 759-72.

[24] Sampson, Thomas, 2012. "Selection into Trade and Wage Inequality," CEP Discussion Papers No. 1152, Centre for Economic Policy Performance.

[25] Sattinger, Michael, 1975. "Comparative Advantage and the Distributions of Earnings and Abilities," *Econometrica 43*(3), 455-68.

[26] Shimer, Robert, 2005. "The Assignment of Workers to jobs in an Economy with Coordination Frictions," *Journal of Political Economy 113(5)*, 996-1025.

[27] Yeaple, Stephen R, 2005. "A Simple Model of Firm Heterogeneity, International Trade, and Wages," *Journal of International Economics 65(1),* 1-20.

Appendix for "Matching and Sorting in a Global Economy"

by

Gene M. Grossman, Elhanan Helpman, and Phillip Kircher

Appendix A

This appendix provides proofs of results stated in the main text.

Proofs for Section 3

First, note that in the system comprising (6)-(9), a proportional increase in the number of managers and workers, $\bar{H}$ and $\bar{L}$, raises H_1 by the same factor of proportionality and leaves the marginal worker q_L^* and the wage anchors w_1 and w_2 unchanged. Therefore, the outputs of the two goods rise equiproportionately, so that the ratio X_1/X_2 does not change. Accordingly, to find the impact of $\bar{H}/\bar{L}$ on X_1/X_2 it suffices to examine the effects of a change in one factor, say $\bar{L}$.

Differentiating the equilibrium system (6)-(9), we obtain

$$\begin{pmatrix} 1 & -1 & s_L\left(q_L^*\right) & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & 0 \\ 0 & \frac{E_2}{1-\gamma_2} & \bar{L}\tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\phi_L\left(q_L^*\right)q_L^* & E_2H_1/H_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\bar{L}\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\phi_L\left(q_L^*\right)q_L^* & -E_1 \end{pmatrix}\begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{H}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_2 \\ -E_1 \end{pmatrix}\hat{\bar{L}} + \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix}\hat{p}_1, \tag{26}$$

where$E_i = H_i\left(\frac{\gamma_i p_i}{w_i}\right)^{\frac{1}{1-\gamma_i}}$, $i = 1, 2$; $H_2 = \bar{H} - H_1$.

Let D_{ho} be the determinant of the matrix on the left-hand side of (26). Then

$$D_{ho} = \bar{L}\phi_L\left(q_L^*\right)q_L^*\frac{w\left(q_L^*\right)}{w_1w_2}H_1\frac{\left(\gamma_2-\gamma_1\right)^2}{\left(1-\gamma_1\right)^2\left(1-\gamma_2\right)^2} + \frac{s_L\left(q_L^*\right)}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left(\gamma_1+\gamma_2\frac{H_1}{H_2}\right)E_1E_2,$$

because (8), (9) and the definition of E_i imply that

$$\begin{aligned} \tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}E_2\frac{H_1}{H_2} - \tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}E_1 &= H_1\left[\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\left(\frac{\gamma_2p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} - \tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\left(\frac{\gamma_1p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}}\right] \\ &= \frac{w\left(q_L^*\right)}{w_1w_2}H_1\frac{\gamma_2-\gamma_1}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}. \end{aligned} \tag{27}$$

It follows that $s_L\left(q_L^*\right) > 0 \Rightarrow D_{ho} > 0$.

We now use (26) to calculate the response of H_1 to an increase in labor supply $\bar{L}$, which yields

$$\hat{H}_1D_{ho} = \frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left[E_1\bar{L}\tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\phi_L\left(q_L^*\right)q_L^* + E_2\bar{L}\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\phi_L\left(q_L^*\right)q_L^* + E_1E_2s_L\left(q_L^*\right)\right]\hat{\bar{L}}.$$

Therefore, given $s_L\left(q_L^*\right) > 0$, an increase in $\bar{L}$ raises the number of managers in sector 1 if and only if $\gamma_1 > \gamma_2$. When H_1 increases, X_1/X_2 does so as well. It follows that the country with relatively more workers produces relatively more of the labor-intensive good. This proves Proposition 2.

Next we calculate the response of the two wage anchors to changes in the price of good 1. From (26) and (27), we obtain

$$\hat{w}_1 D_{ho} = \bar{L}\phi_L(q_L^*)\, q_L^* \frac{1}{1-\gamma_1}\frac{w(q_L^*)}{w_1 w_2} H_1 \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)}\hat{p}_1 + s_L(q_L^*)\frac{E_1 E_2}{(1-\gamma_1)(1-\gamma_2)}\left(1+\gamma_2\frac{H_1}{H_2}\right)\hat{p}_1,$$

$$\hat{w}_2 D_{ho} = \bar{L}\phi_L(q_L^*)\, q_L^* \frac{1}{1-\gamma_1}\frac{w(q_L^*)}{w_1 w_2} H_1 \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)}\hat{p}_1 - s_L(q_L^*)\frac{E_1 E_2}{1-\gamma_1}\frac{H_1}{H_2}\hat{p}_1.$$

It follows that $s_L(q_L^*) > 0 \Rightarrow (\hat{w}_1 - \hat{w}_2)/\hat{p}_1 > 0$, which implies that $\hat{w}_1 > \hat{w}_2$ when $\hat{p}_1 > 0$, as stated in part (i) of Proposition 4. We also calculate the response of the managers' salary, using (10) with $i = 2$. We find

$$\hat{r} = -\frac{\gamma_2}{1-\gamma_2}\hat{w}_2.$$

Evidently, the managers' salary moves in the opposite direction to the wage anchor in sector 2.

Now consider the cases discussed in parts (ii)-(iv) of Proposition 4. In case (ii) we have $\gamma_1 \approx \gamma_2$ and therefore

$$\hat{w}_1 \approx \frac{\gamma_1^{-1} + \frac{H_1}{H_2}}{1+\frac{H_1}{H_2}}\hat{p}_1,$$

$$\hat{w}_2 \approx -\frac{(1-\gamma_1)\frac{H_1}{H_2}}{\gamma_1\left(1+\frac{H_1}{H_2}\right)}\hat{p}_1,$$

$$\hat{r} \approx \frac{\frac{H_1}{H_2}}{1+\frac{H_1}{H_2}}\hat{p}_1.$$

It follows that $\hat{w}_1 > \hat{p}_1 > \hat{r} > 0 > \hat{w}_2$, which proves part (ii) of the proposition. In cases (iii) and (iv), we have $s_L(q_L^*) \approx 0$, which implies

$$\hat{w}_1 \approx \hat{w}_2 \approx \frac{1-\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1,$$

$$\hat{r} = -\frac{\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1.$$

Therefore, if $\gamma_1 > \gamma_2$ then $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}$ and if $\gamma_1 < \gamma_2$ then $\hat{r} > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$, which proves parts (iii) and (iv).

We now consider the impact of a rightward shift of the density function $\phi(q_L)$, as defined in (12). To perform these comparative statics, we can equivalently hold the distribution of types constant but endow a worker of type q_L with λq_L units of ability. In each sector, the demand for efficiency units of labor must equal the supply. A worker in sector i of type q_L provides $\tilde{\psi}_i(\lambda q_L)^{1/\gamma_i}$ units of efficiency labor. The labor-market clearing conditions should now be written as

$$H_1\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} = \bar{L}\int_{q_L^*}^{q_{L\max}} \tilde{\psi}_1(\lambda q)^{1/\gamma_1}\phi_L(q)\, dq \tag{28}$$

and

$$(\bar{H} - H_1)\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} = \bar{L}\int_{q_{L\min}}^{q_L^*} \tilde{\psi}_2(\lambda q)^{1/\gamma_2}\phi_L(q)\, dq. \tag{29}$$

A worker of type q_L employed in sector i earns the salary

$$w(q_L) = w_i \tilde{\psi}_i (\lambda q_L)^{1/\gamma_i} \ ,$$

so wage continuity at the marginal worker q_L^* requires

$$w_1 \tilde{\psi}_1 (\lambda q_L^*)^{1/\gamma_1} = w_2 \tilde{\psi}_2 (\lambda q_L^*)^{1/\gamma_2} . \tag{30}$$

Finally, profits for a firm in industry i that hires workers with index q_L are

$$\tilde{\pi}_i (q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left[\tilde{\psi}_i (\lambda q_L)\right]^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r$$

and free entry in both sectors implies

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}} . \tag{31}$$

Equations (28) - (31) determine q_L^*, H_1, w_1 and w_2.

Now we define $Q_L^* = \lambda q_L^*$ and totally differentiate the equilibrium system (evaluated at $\lambda = 1$), which yields

$$\begin{pmatrix} 1 & -1 & s_L(q_L^*) & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & 0 \\ 0 & \frac{E_2}{1-\gamma_2} & \bar{L}\tilde{\psi}_2 (q_L^*)^{1/\gamma_2} \phi_L(q_L^*) q_L^* & E_2 H_1/H_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\bar{L}\tilde{\psi}_1 (\lambda q_L^*)^{1/\gamma_1} \phi_L(q_L^*) q_L^* & -E_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{Q}_L^* \\ \hat{H}_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -\frac{E_2}{\gamma_2}\bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*) + \bar{L}\tilde{\psi}_2 (q_L^*)^{1/\gamma_2} \phi_L(q_L^*) q_L^* \\ -\frac{E_1}{\gamma_1}\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) - \bar{L}\tilde{\psi}_1 (q_L^*)^{1/\gamma_1} \phi_L(q_L^*) q_L^* \end{pmatrix} \hat{\lambda},$$

where $\bar{\varepsilon}_{\tilde{\psi}_i}(q_L^*)$ is a weighted average of the elasticities $\varepsilon_{\tilde{\psi}_i}(q_L)$ in sector i, with weights

$$v_i(q_L) = \frac{\tilde{\psi}_i (q_L)^{1/\gamma_i} \phi_L(q_L)}{\int_{q_L \in Q_{Li}} \tilde{\psi}_i (q_L)^{1/\gamma_i} \phi_L(q_L) \, dq_L}, \quad i = 1, 2.$$

Therefore,

$$\hat{H}_1 D_{ho} (1-\gamma_1)(1-\gamma_2) = E_1 E_2 s_L(q_L^*) \left[\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) - \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)\right] \hat{\lambda}$$
$$+ \bar{L}\phi_L(q_L^*) q_L^* \left[\tilde{\psi}_1 (q_L^*)^{1/\gamma_1} E_2 + \tilde{\psi}_2 (q_L^*)^{1/\gamma_2} E_1\right] \left[\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) - \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)\right] \hat{\lambda}.$$

It follows that, given $s_L(q_L^*) > 0$, an increase in λ raises H_1 if and only if $\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) > \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)$. Moreover, $\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) > \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)$ if $\varepsilon_{\tilde{\psi}_1}(q_L') > \varepsilon_{\tilde{\psi}_2}(q_L'')$ for all $q_L', q_L'' \in S_L^A \cup S_L^B$ and $\bar{\varepsilon}_{\tilde{\psi}_1}(q_L^*) < \bar{\varepsilon}_{\tilde{\psi}_2}(q_L^*)$ if $\varepsilon_{\tilde{\psi}_1}(q_L') <$

$\varepsilon_{\tilde{\psi}_2}(q''_L)$ for all $q'_L, q''_L \in S_L^A \cup S_L^B$. This proves Proposition 3.

Proofs for Section 4

Denote by $m_i(q_H)$ the solution set to problem (15). Because S_L and S_H are compact, $m_i(q_H)$ is upper hemicontinuous (because $\tilde{\pi}_i(q_L, q_H)$ is a continuous function), and $m_i(q_H)$ is closed-valued, the graph

$$G_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in S_H]$$

is closed. The matching correspondence satisfies

$$m(q_H) = \begin{cases} m_1(q_H) & \text{for } q_H \in Q_{H1} \\ m_2(q_H) & \text{for } q_H \in Q_{H2} \end{cases}$$

and the equilibrium allocation graph in sector i is

$$M_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in Q_{Hi}] \subseteq G_i.$$

Since $Q_{Hi} \subseteq S_H$, the graph M_i is also closed.

Now consider a connected subset $M_i^n \subseteq M_i$:

$$M_i^n = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in [q_{H1}, q_{H2}] \subseteq Q_{Hi}].$$

Since M_i is a closed graph, such a subset exists and there exists an interval $[q_{L1}, q_{L2}]$, $q_{L2} > q_{L1}$, that satisfies both (i) $m_i(q_H) \in [q_{L1}, q_{L2}]$ for all $q_H \in [q_{H1}, q_{H2}]$ and (ii) for every point $q_L \in [q_{L1}, q_{L2}]$ there exists a managerial ability level $q_H \in [q_{H1}, q_{H2}]$ satisfying $q_L \in m_i(q_H)$. This means that, in M_i^n, workers of ability $[q_{L1}, q_{L2}]$ are matched with managers of ability $[q_{H1}, q_{H2}]$ and all workers and managers have matches. Then, as Eeckhout and Kircher (2012) have shown, strict log supermodularity of $\psi_i(\cdot)$ ensures strict positive assortative matching (PAM) between the factors allocated to sector i. It follows that $m_i(q_H)$ is a continuous and strictly increasing function in the interior of $[q_{H1}, q_{H2}]$. M_i consists of a union of connected sets, $M_i = \cup_{n \in N_i} M_i^n$, such that $m_i(q_H)$ is continuous and strictly increasing in each such set and $m_i(q_H)$ jumps upwards between them.

We now establish the differentiability of $w(\cdot)$ in $M_i^{n,int}$.[40] Let $m^{-1}(\cdot)$ be the inverse of the sectoral matching function in $M_i^{n,int}$. Since $m(\cdot)$ is continuous and strictly increasing in $M_i^{n,int}$, this inverse exists. Now consider an interval $[q'_L, q'_L + dq_L) \in M_i^{n,int}$. The zero-profit condition (16) implies

$$w(q'_L) = \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q'_L), q'_L\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}(q'_L)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w(q'_L + dq_L) \geq \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q'_L), q'_L + dq_L\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}(q'_L)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

[40] This proof is similar to the proof of differentiability of the wage function in Sampson (2012).

Together, these expressions imply

$$w\left(q_L'+dq_L\right)\geq w\left(q_L'\right)\left\{\frac{\psi_i\left[m^{-1}\left(q_L'\right),q_L'+dq_L\right]}{\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]}\right\}^{\frac{1}{\gamma_i}}. \tag{32}$$

Similarly, (16) implies

$$w\left(q_L'+dq_L\right)=\bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}}p_i^{\frac{1}{\gamma_i}}\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'+dq_L\right]^{\frac{1}{\gamma_i}}r\left[m^{-1}\left(q_L'+dq_L\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w\left(q_L'\right)\geq\bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}}p_i^{\frac{1}{\gamma_i}}\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]^{\frac{1}{\gamma_i}}r\left[m^{-1}\left(q_L'+dq_L\right)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

Together, these expressions imply

$$w\left(q_L'\right)\geq w\left(q_L'+dq_L\right)\left\{\frac{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]}{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'+dq_L\right]}\right\}^{\frac{1}{\gamma_i}}. \tag{33}$$

Inequalities (32) and (33) jointly imply

$$\frac{w\left(q_L'\right)}{\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}\left[\frac{\psi_i\left[m^{-1}\left(q_L'\right),q_L'+dq_L\right]^{\frac{1}{\gamma_i}}-\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L}\right]\leq\frac{w\left(q_L'+dq_L\right)-w\left(q_L'\right)}{dq_L}$$
$$\leq\frac{w\left(q_L'\right)}{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]^{\frac{1}{\gamma_i}}}\left[\frac{\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'+dq_L\right]^{\frac{1}{\gamma_i}}-\psi_i\left[m^{-1}\left(q_L'+dq_L\right),q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L}\right].$$

Since, by Assumption 3, the productivity function is continuous, strictly increasing, and differentiable, and since the inverse of the sectoral matching function is continuous and strictly increasing in this range, taking the limit as $dq_L\to 0$ implies that the derivative of $w(\cdot)$ at q_L' exists and

$$\frac{dw\left(q_L'\right)}{dq_L}=\frac{w\left(q_L'\right)}{\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}\frac{\partial\psi_i\left[m^{-1}\left(q_L'\right),q_L'\right]^{\frac{1}{\gamma_i}}}{\partial q_L}.$$

Similar arguments can be used to show that the salary function is differentiable.

We now prove Proposition 6 by contradiction. (Proposition 5 can be proved similarly.) To this end, suppose that the inequality condition holds, but the equilibrium is such that there are managers employed in sector j who have greater ability than some managers employed in sector i. In such circumstances, there exists an ability level $\tilde{q}_H$ at one of the boundaries between Q_{Hi} and Q_{Hj} such that managers with ability in $\left(\tilde{q}_H-\varepsilon_i,\tilde{q}_H\right)\subset Q_{Hi}^{int}$ are employed in sector i and managers with ability $\left(\tilde{q}_H,\tilde{q}_H+\varepsilon_j\right)\subset Q_{Hj}^{int}$ are employed in sector j, for $\varepsilon_i>0$ and $\varepsilon_j>0$ small enough. Moreover, the equilibrium conditions (16)-(18) are satisfied, the matching function $m(q_H)$ is continuous at Q_{Hi}^{int} and Q_{Hj}^{int} close to $\tilde{q}_H$ (but can be discontinuous at the boundary point between these sets), the wage function $w(q_L)$ is continuous and increasing in S_L and differentiable in Q_{Li}^{int} and Q_{Lj}^{int}, and the salary function $r(q_H)$ is continuous and increasing in S_H and differentiable in Q_{Hi}^{int} and Q_{Hj}^{int}.

Now recall the continuous profit function $\Pi_i(q_H)$ defined in (15). In equilibrium, $\Pi_i(q_H) = 0$ for all $q_H \in Q_{Hi}$, but the maximal profits $\Pi_i(q_H)$ may differ from zero for $q_H \notin Q_{Hi}$. Therefore $\Pi_i(q_H) = 0$ for all $q_H \in (\tilde{q}_H - \varepsilon_i, \tilde{q}_H)$ and, by continuity, $\lim_{q_H \nearrow \tilde{q}_H} \Pi_i(q_H) = 0$.

Next consider the profits that would accrue to an entrepreneur that hires a manager with ability $\tilde{q}_H + \varepsilon$ in order to produce good i, where $\varepsilon < \varepsilon_j$. Choosing workers so as to maximize profits, this entrepreneur earns $\Pi_i(\tilde{q}_H + \varepsilon) \geq \pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$, where $m\left(\tilde{q}_H^-\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H - \varepsilon)$ and $\lim_{\varepsilon \searrow 0} \Pi_i(\tilde{q}_H + \varepsilon) = \lim_{\varepsilon \searrow 0} \pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] = 0$. The first-order approximation to $\pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$ is

$$\pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \approx \varepsilon \pi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right],$$

where $\pi_{iH}(\cdot)$ is the partial derivative of $\pi_i(\cdot)$ with respect to q_H. This derivative exists because the salary function is differentiable in Q_{Hj}^{int}, and

$$\begin{aligned}
&\pi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \\
&= \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]^{\frac{1}{1-\gamma_i}} w\left[m\left(\tilde{q}_H^-\right)\right]^{-\frac{\gamma_i}{1-\gamma_i}} \frac{\psi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]} - r'(\tilde{q}_H + \varepsilon) \\
&= \left\{\frac{\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{\psi_i\left[\tilde{q}_H, m\left(\tilde{q}_H^-\right)\right]}\right\}^{\frac{1}{1-\gamma_i}} r\left(\tilde{q}_H^-\right) \frac{\psi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]} - r'(\tilde{q}_H + \varepsilon),
\end{aligned}$$

where the last equality uses the free-entry condition (16), which applies to sector 1 at points in Q_{Hi}^{int} in the conjectured equilibrium, and $r\left(\tilde{q}_H^-\right) = r(\tilde{q}_H)$ due to the continuity of the salary function. Since $\tilde{q}_H + \varepsilon \in Q_{Hj}^{int}$, condition (18) implies

$$\lim_{\varepsilon \searrow 0} \pi_{iH}\left[\tilde{q}_H + \varepsilon, m_i\left(\tilde{q}_H^-\right)\right] = r(\tilde{q}_H)\left\{\frac{\psi_{iH}\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]} - \frac{\psi_{jH}\left[q_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_j)\psi_j\left[q_H, m\left(\tilde{q}_H^+\right)\right]}\right\},$$

where $m\left(\tilde{q}_H^+\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H + \varepsilon)$. It now follows from supposition of Proposition 6 that the right-hand side of this equation is strictly positive irrespective of the values of $m_i\left(\tilde{q}_H^-\right)$ and $m\left(\tilde{q}_H^+\right)$, and therefore that $\pi_{iH}\left[\tilde{q}_H + \varepsilon, m_i\left(\tilde{q}_H^-\right)\right] > 0$ for ε small enough, which contradicts the zero-profit condition as profits rise above zero This contradicts the supposition that in equilibrium there are managers employed in sector j who are more able than some managers employed in sector i. Consequently, every manager in sector i has greater ability than any manager employed in sector j. This completes the proof.

Next we prove Proposition 7. Suppose that the inequality conditions in Proposition 7 hold but the equilibrium is such that there exist managers in sector 2 who are more able than some managers in sector 1. In such circumstances, there exists an ability $\tilde{q}_H$ at one of the boundary points between Q_{H1} and Q_{H2} such that managers of ability $\tilde{q}_H - \varepsilon_1$ are employed in sector 1 and managers of ability $\tilde{q}_H + \varepsilon_2$ are employed in sector 2 for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. Let $m(\tilde{q}_H^-) = \lim_{q_H \nearrow \tilde{q}_H} m(q_H)$ and $m(\tilde{q}_H^+) = \lim_{q_H \searrow \tilde{q}_H} m(q_H)$ Then

$$\lim_{\varepsilon \to 0} \pi_{iH}\left[\tilde{q}_H + \varepsilon, m(q_H^-)\right] = r(\tilde{q}_H)\left[\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{(1-\gamma_1)\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} - \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_2)\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}\right], \qquad (34)$$

which we derive in the same way as in the proof of Proposition 6. Under the supposition that the managers to the left of $\tilde{q}_H$ sort into sector 1 and those to the right of $\tilde{q}_H$ sort into sector 2 the partial derivative in

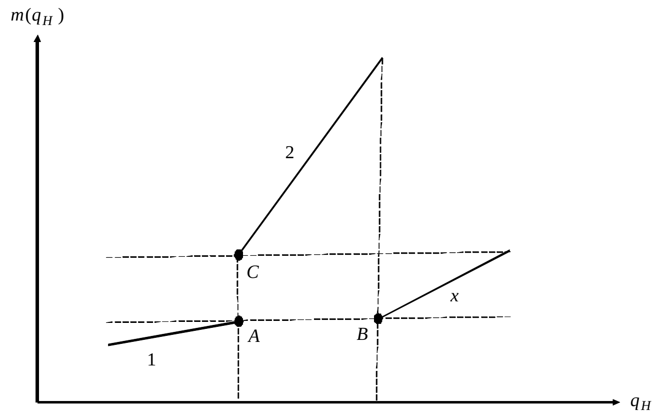

Figure 11: Matching function with discontinuity

(34) cannot be positive and therefore

$$\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{(1-\gamma_1)\,\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} \leq \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_2)\,\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}.$$

In view of the first inequality in Proposition 7 and the strict log supermodularity of the productivity function, this inequality implies $m\left(\tilde{q}_H^+\right) > m\left(\tilde{q}_H^-\right)$. That is, the matching function is discontinuous at $\tilde{q}_H$ and it jumps upwards there. As a result, there must exist an ability level for workers $\check{q}_L \in \left[m\left(\tilde{q}_H^-\right), m\left(\tilde{q}_H^+\right)\right]$ such that workers in the range $(\check{q}_L - \breve{\varepsilon}_1, \check{q}_L)$ are employed in sector 1 and workers in the range $(\check{q}_L, \check{q}_L + \breve{\varepsilon}_2)$ are employed in sector 2, for $\breve{\varepsilon}_1$ and $\breve{\varepsilon}_2$ small enough. Due to the upward jump of the matching function and due to PAM in each sector, in this range of worker types the ability of managers matched with workers in sector 1 must be strictly greater than the ability of managers matched with workers in sector 2. This is illustrated in Figure 11. At point A, we have $q_H = \tilde{q}_H$ and the matching function exhibits an upward jump from point A to C. The supposition is that managers to the left of A sort into sector 1 and managers to the right of A sort into sector 2, as illustrated in the figure. Clearly, workers with ability between points A and C must be matched with managers in some sector. Segment x illustrates a possible matching of these workers with high-ability managers. It is not possible for x to be sector 2, however, because this would imply non-monotonic matching in this sector, which is ruled out by the strict log supermodularity of the productivity function there. So x must be sector 1. In this case, $\check{q}_L$ is the ability of workers at point C. Workers with ability just below C are employed in sector 1 and workers with ability just above C are employed in sector 2. Evidently, the ability of managers with whom these workers are matched in sector 1 is higher than the ability of managers with whom their slightly better peers are matched in sector 2. It can be seen from the figure that a similar outcome obtains if the matching along x is to the left of point A, except that in this case x stands for sector 2 and $\check{q}_L$ is the ability of workers at point A. Evidently, in this case too, at points around $\check{q}_L$ the ability of managers matched with workers in sector 1 is higher than the ability of managers matched with workers in sector 2.

In short, consider the inverse function $m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \breve{\varepsilon}_1, \check{q}_L)$; this inverse exists in the specified range because $m_1(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H - \varepsilon, \tilde{q}_H)$ for ε small enough. Similarly, consider the inverse function $m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$; this inverse also exists in the specified range because $m_2(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H, \tilde{q}_H + \varepsilon)$ for ε small enough. Moreover, under the supposition of our sorting pattern $m^{-1}(q_L) = m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $m^{-1}(q_L) = m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$ and the argument in the previous paragraph showed that $m^{-1}(q_L) = m_1^{-1}(q_L) > m^{-1}(q_L') = m_2^{-1}(q_L')$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $q_L' \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$. Taking limits as $\check{\varepsilon}_1, \check{\varepsilon}_2 \searrow 0$, this implies that $m^{-1}\left(\check{q}_L^-\right) > m^{-1}\left(\check{q}_L^+\right)$.

Next, following steps similar to those used in the proof of Proposition 6, which considered the response of profits to variations in the ability of managers at points around $\tilde{q}_H$, an analysis of the response of profits to variations in the ability of workers at points around $\check{q}_L$ establishes that a necessary condition for optimality is

$$\frac{\psi_{1L}\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]}{\gamma_1 \psi_1 \left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]} \leq \frac{\psi_{2L}\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}{\gamma_2 \psi_2 \left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}.$$

In view of the second inequality in Proposition 7 and the strict log supermodularity of the productivity function, this inequality implies $m^{-1}\left(\check{q}_L^+\right) = m_2^{-1}\left(\check{q}_L^+\right) > m_1^{-1}\left(\check{q}_L^-\right) = m^{-1}\left(\check{q}_L^-\right)$, which contradicts the above established result that $m_1^{-1}\left(\check{q}_L^-\right) > m_2^{-1}\left(\check{q}_L^+\right)$. It follows that the best managers sort into sector 1. By symmetrical arguments the best workers also sort into sector 1.

Matching and Factor Prices Among a Group of Workers and Managers

In order to prove the remaining propositions in the main text, we need to understand how matching within an allocation set and the wages and salaries of workers and managers in the set respond to changes in factor endowments, the price of the output produced by these factors, and the boundaries of workers' and managers' abilities.

Suppose that some sector employs workers and managers whose abilities form the intervals $S_L = [q_{La}, q_{Lb}]$ and $S_H = [q_{Ha}, q_{Hb}]$. To simplify notation, we drop the sectoral index i and denote q_H by q, and we consider the following industry equilibrium conditions:

$$r(q) = \bar{\gamma} p^{\frac{1}{1-\gamma}} \psi[q, m(q)]^{\frac{1}{1-\gamma}} w[m(q)]^{-\frac{\gamma}{1-\gamma}}, \quad \bar{\gamma} = \gamma^{\frac{\gamma}{1-\gamma}}(1-\gamma) \tag{35}$$

$$\frac{\psi_L[q, m(q)]}{\gamma \psi[q, m(q)]} = \frac{w'[m(q)]}{w[m(q)]}, \tag{36}$$

$$\bar{H} \frac{\gamma r(q)}{(1-\gamma) w[m(q)]} \phi_H(q) = \bar{L} \phi_L[m(q)] m'(q), \tag{37}$$

and the boundary conditions,

$$\begin{aligned} m(q_{Hz}) &= q_{Lz}, \quad z = a, b; \\ q_{Lb} &> q_{La} > 0, \quad q_{Hb} > q_{Ha} > 0. \end{aligned} \tag{38}$$

Equation (35) is taken from (16), (36) is taken from (17) and (37) is taken from (19). We seek to characterize the solution for the three functions, $w(\cdot)$, $r(\cdot)$ and $m(\cdot)$.

We use (35) and (36) to obtain

$$\ln r(q_H) - \ln r(q_{H0}) = \int_{q_{H0}}^{q_H} \frac{\psi_H[x, m(x)]}{(1-\gamma)\psi[x, m(x)]} dx, \quad \text{for } q_H, q_{H0} \in S_H, \tag{39}$$

$$\ln w(q_L) - \ln w(q_{L0}) = \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu(x), x]}{\gamma\psi[\mu(x), x]} dx, \quad \text{for } q_L, q_{L0} \in S_L, \tag{40}$$

where $\mu(\cdot)$ is the inverse of $m(\cdot)$. We substitute (35) into (37) to obtain

$$\begin{aligned} \frac{1}{1-\gamma}\ln w[m(q)] \quad &= \quad \frac{1}{1-\gamma}\ln\gamma + \ln\left(\frac{\bar{H}}{\bar{L}}\right) + \frac{1}{1-\gamma}\ln p \\ &\quad + \frac{1}{1-\gamma}\ln\psi[q, m(q)] + \log\phi_H(q) - \log\phi_L[m(q)] - \log m'(q). \end{aligned} \tag{41}$$

The differential equations (36) and (41) together with the boundary conditions (38) uniquely determine the solution of $w(\cdot)$ and $m(\cdot)$ when the productivity function $\psi(\cdot)$ is twice continuously differentiable and the density functions $\phi_F(\cdot)$, $F = H, L$, are continuously differentiable.

By differentiating (41) and substituting (36) into the result, we generate a second-order differential equation for the matching function,

$$\frac{m''(q)}{m'(q)} = \frac{\psi_H[q, m(q)]}{(1-\gamma)\psi_L[q, m(q)]} - \frac{\psi_L[q, m(q)]\, m'(q)}{\gamma\psi[q, m(q)]} + \frac{\phi'_H(q)}{\phi_H(q)} - \frac{\phi'_L[m(q)]\, m'(q)}{\phi_L[m(q)]}. \tag{42}$$

Given boundary conditions $m(q_0) = q_{L0}$, $m'(q_0) = t_0 > 0$, this differential equation has a unique solution, which may or may not satisfy the boundary conditions (38). The solution to the original matching problem is found by identifying a value t_a such that $m(q_{Ha}) = q_{La}$ and $m'(q_{Ha}) = t_a$ yield a solution that satisfies the second boundary condition $m(q_{Hb}) = q_{Lb}$. Note that this solution depends neither on the price p nor on the factor endowments $\bar{H}$ and $\bar{L}$. Therefore, changes in these variables do no affect the matching function, but they change all wages and salaries proportionately, as can be seen from (41), and (35). We have

Lemma 1 *(i) The matching function $m(\cdot)$ does not depend on $(p, \bar{H}, \bar{L})$. (ii) An increase in the price p, $\hat{p} > 0$, raises the wage and salary schedules proportionately by $\hat{p}$. (iii) An increase in $\bar{H}/\bar{L}$ such that $\hat{H} - \hat{L} = \hat{\eta} > 0$ raises the wage schedule proportionately by $(1-\gamma)\hat{\eta}$ and reduces the salary schedule proportionately by $\gamma\hat{\eta}$.*

We now prove several lemmas that are used in the main analysis.

Lemma 2 *Let $[m_\varkappa(q), w_\varkappa(q_L)]$ and $[m_\varrho(q), w_\varrho(q_L)]$ be solutions to the differential equations (36) and (41), each for different boundary conditions (38), such that $m_\varkappa(q_0) = m_\varrho(q_0) = q_{L0}$ and $m'_\varrho(q_0) > m'_\varkappa(q_0)$ for $q_0 \in S_{H\varkappa} \cap S_{H\varrho}$. Then $m_\varrho(q) > m_\varkappa(q)$ for all $q > q_0$ and $m_\varrho(q) < m_\varkappa(q)$ for all $q < q_0$ in the overlapping range of abilities.*

Proof. Consider $q > q_0$ and suppose that, contrary to the claim, there exists a $q_1 > q_0$ such that $m_\varrho(q_1) \leq m_\varkappa(q_1)$. Then differentiability of $m_\iota(\cdot)$, $\iota = \varkappa, \varrho$, implies that there exists $q_2 > q_0$ such that $m_\varrho(q_2) = m_\varkappa(q_2)$, $m_\varrho(q) > m_\varkappa(q)$ for all $q \in (q_0, q_2)$ and $m'_\varrho(q_2) < m'_\varkappa(q_2)$. This also implies $\mu_\varrho(x) < \mu_\varkappa(x)$

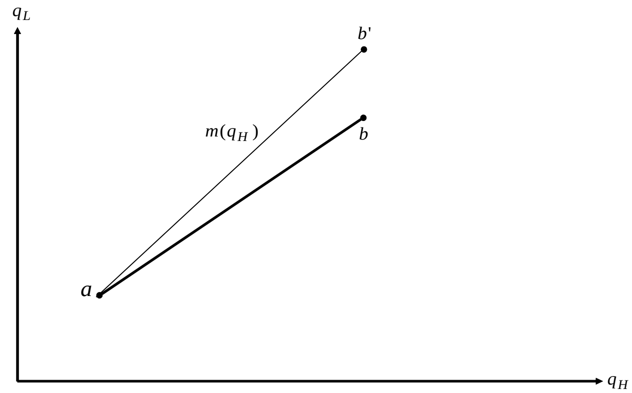

Figure 12: Shift in the matching function when q_L^b rises to $q_L^{b'}$

for all $x \in (m_\varrho(q_0), m_\varrho(q_2))$, where $\mu_\iota(\cdot)$ is the inverse of $m_\iota(\cdot)$. Under these conditions (41) implies $w_\varrho[m_\varrho(q_0)] < w_\varkappa[m_\varrho(q_0)]$ and $w_\varrho[m_\varrho(q_2)] > w_\varkappa[m_\varrho(q_2)]$, and therefore

$$w_\varkappa[m_\varrho(q_2)] - w_\varkappa[m_\varrho(q_0)] < w_\varrho[m_\varrho(q_2)] - w_\varrho[m_\varrho(q_0)].$$

On the other hand, (40) implies

$$\ln w_\iota[m_\varrho(q_2)] - \ln w_\iota[m_\varrho(q_0)] = \int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\iota(x), x]}{\gamma\psi[\mu_\iota(x), x]} dx, \quad \iota = \varkappa, \varrho.$$

Together with the previous inequality, this gives

$$\int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\varkappa(x), x]}{\psi[\mu_\varkappa(x), x]} dx < \int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\varrho(x), x]}{\psi[\mu_\varrho(x), x]} dx.$$

Note, however, that strict log supermodularity of $\psi(\cdot)$ and $\mu_\varrho(x) < \mu_\varkappa(x)$ for all $x \in (m_\varrho(q_0), m_\varrho(q_2))$ imply the reverse inequality, a contradiction. It follows that $m_\varrho(q) > m_\varkappa(q)$ for all $q > q_0$. A similar argument shows that $m_\varrho(q) < m_\varkappa(q)$ for all $q < q_0$. ■

The key implication of this lemma is that changes in the boundary conditions (38) shift the matching function in such a way as to generate at most one point in common with the original matching function. We next show how the matching function and wage function respond to the boundary conditions. To this end, re-consider Figure 4 in the main text. Let the thick curve between points a and b represent the solution to the matching function when points a and b are the boundary points (38). Now consider the shift of the equilibrium matching function in response to a rise in q_{Lb}; that is, the end point b shifts upward to b'. Since point a is common to the old and new matching function, Lemma 2 implies that the two curves can have no additional points in common, which implies that the new inverse matching function—represented by the thin curve between points a and b'—is everywhere above the old one. It follows that an increase in q_{Lb} increases the ability of workers matched with every manager except for the least able manager. Other shifts in the boundary points can be analyzed in similar fashion to establish

Lemma 3 *(i) $dm(q_H)/dq_{La} > 0$ for all $q_H < q_{Hb}$ and $d\mu(q_L)/dq_{La} < 0$ for all $q_L < q_{Lb}$; (ii) $dm(q_H)/dq_{Lb} > 0$ for all $q_H > q_{Ha}$ and $d\mu(q_L)/dq_{Lb} < 0$ for all $q_L > q_{La}$; (iii) $d\mu(q_L)/dq_{Ha} > 0$ for all $q_L < q_{Lb}$ and $dm(q_H)/dq_{Ha} < 0$ for all $q_H < q_{Hb}$; and (iv) $d\mu(q_L)/dq_{Hb} > 0$ for all $q_L > q_{La}$ and $dm(q_H)/dq_{Hb} < 0$ for all $q_H > q_{Ha}$.*

The rule that emerges from this lemma is that an improvement in the ability of workers at a boundary of S_L improves the quality of the matches for all the managers (except those at the other boundary) and deteriorates the quality of the matches for all the workers (except those at the other boundary). Similarly, an improvement in the ability of managers at a boundary of S_H improves the quality of the matches for all workers (except those at the other boundary) and deteriorates the quality of the matches for all the managers (except those at the other boundary).

Next consider changes in a boundary (q_{Hz}, q_{Lz}), $z = a, b$. For concreteness, suppose that (q_{Hb}, q_{Lb}) changes. Then the new matching function coincides with the old one at the other boundary point, (q_{Ha}, q_{La}), which has not changed. In this case, Lemma 2 implies that either the two matching functions coincide in the overlapping range of abilities or one is above the other everywhere except for at (q_{Ha}, q_{La}). A similar argument applies to changes in (q_{Ha}, q_{La}). We therefore have:

Lemma 4 *In response to a shift in a single boundary (q_{Hz}, q_{Lz}), $z = a, b$, either the new matching functions coincide with the old matching function in the overlapping range of abilities or one matching function is above the other everywhere except for at the opposite boundary point.*

We next discuss the impact of boundaries on wages and salaries. We focus on wages, but note that if a shift in boundaries raises the wage of workers with ability q_L then it must reduce the salary of managers teamed with these workers. This can be seen from (35) by noting that a change in boundaries has no impact on $r(\cdot)$ through an induced shift in the matching function due to the first-order condition (36) (a version of the Envelope Theorem). Therefore the change in salary $r(q)$ is driven by the change in wages of workers matched with managers of ability q. We record this result in

Lemma 5 *Suppose that the boundaries (q_{Hz}, q_{Lz}), $z = a, b$, change and that, as a result, $w(q_L)$ rises for some q_L such that q_L and $q = m^{-1}(q_L)$ are in the overlapping range of abilities of the old and new boundaries. Then $r(q)$ declines.*

For the subsequent analysis the following lemma is useful:

Lemma 6 *Let $[m_\varkappa(q), w_\varkappa(q_L)]$ and $[m_\varrho(q), w_\varrho(q_L)]$ be solutions to the differential equations (36) and (41), each for different boundary conditions (38), such that $m_\varkappa(q_0) = m_\varrho(q_0) = q_{L0}$ and $m'_\varrho(q_0) > m'_\varkappa(q_0)$ for some $q_0 \in S_{L\varkappa} \cap S_{L\varrho}$, and let $r_\varrho(q)$ and $r_\varkappa(q)$ be the corresponding solutions to (35). Then $w_\varrho(q_L) < w_\varkappa(q_L)$ and $r_\varrho(q) > r_\varkappa(q)$ in the overlapping range of abilities.*

Proof. From Lemma 2 we know that $m_\varrho(q) > m_\varkappa(q)$ for all $q > q_0$ and $m_\varrho(q) < m_\varkappa(q)$ for all $q < q_0$ in the overlapping range of abilities and $\mu_\varrho(x) < \mu_\varkappa(x)$ for all $x > q_{L0}$ and $\mu_\varrho(x) > \mu_\varkappa(x)$ for all $x < q_{L0}$ in the overlapping range of abilities. Moreover, $m'_\varrho(q_0) > m'_\varkappa(q_0)$ and (41) imply

$$\ln w_\varkappa(q_{L0}) > \ln w_\varrho(q_{L0})$$

while (40) implies

$$\ln w_\iota(q_L) - \ln w_\iota(q_{L0}) = \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_\iota(x), x]}{\gamma\psi[\mu_\iota(x), x]} dx, \quad \iota = \varkappa, \varrho.$$

Together, these inequalities imply

$$\begin{aligned} \ln w_\varkappa(q_L) - \ln w_\varrho(q_L) &> \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_\varkappa(x), x]}{\gamma\psi[\mu_\varkappa(x), x]} dx - \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_\varrho(x), x]}{\gamma\psi[\mu_\varrho(x), x]} dx \\ &= \int_{q_L}^{q_{L0}} \frac{\psi_L[\mu_\varrho(x), x]}{\gamma\psi[\mu_\varrho(x), x]} dx - \int_{q_L}^{q_{L0}} \frac{\psi_L[\mu_\varkappa(x), x]}{\gamma\psi[\mu_\varkappa(x), x]} dx. \end{aligned}$$

For $q_L > q_{L0}$ the right-hand side of the first line is positive due to the strict log supermodularity of the productivity function and $\mu_\varrho(x) < \mu_\varkappa(x)$ for all $x > q_{L0}$, and the second line also is positive for $q_L < q_{L0}$ due to the strict log supermodularity of the productivity function and $\mu_\varrho(x) > \mu_\varkappa(x)$ for all $x < q_{L0}$. It follows that $w_\varkappa(q_L) > w_\varrho(q_L)$ for all q_L in the overlapping range of abilities. A similar argument establishes that $r_\varkappa(q) < r_\varrho(q)$ for all q in the overlapping range of abilities. ■

This lemma, together with Lemma 4, have straightforward implications for the impact of boundary points on the wage and salary functions.

Corollary 1 *Suppose that the lower boundary (q_{Ha}, q_{La}) changes and the matching function shifts upwards as a result. Then salaries decline and wages rise in the overlapping range of abilities. The converse holds when the matching function shifts downwards.*

Corollary 2 *Suppose that the upper boundary (q_{Hb}, q_{Lb}) changes and the matching function shifts upwards as a result. Then salaries rise and wages decline in the overlapping range of abilities. The converse holds when the matching function shifts downwards.*

Not only do wages and salaries shift in a predictable way in response to a shift in a boundary point, the inequality of wages and of salaries also change in predictable ways. From (40) we see that a change in boundaries that shifts upwards the matching function reduces wage inequality, because for every two ability levels the ratio of the wage of a high-ability worker to the wage of a low-ability worker declines for all types in the overlapping range. For salaries it is the opposite, as one can see from (39). We therefore have

Lemma 7 *Suppose that the matching function shifts upwards in response to a shift in the boundaries (38). Then wage inequality narrows and salary inequality widens. The opposite is true when the matching function shifts downwards.*

General Equilibrium

Consider a two-sector economy in which the most-able workers are employed in one sector and the least-able workers are employed in the other sector, and similarly for managers. In such circumstances, the equilibrium can take one of two forms: either the highest-ability workers and highest-ability managers are employed in the same sector and the lowest-ability workers and lowest-ability managers are employed in the other, which we designated as an HH/LL equilibrium, or the highest-ability workers and lowest-ability managers are employed in one sector and the lowest-ability workers and highest-ability managers are employed in the other, which we designated as an HL/LH equilibrium. Our first result is

Lemma 8 *Suppose that the economy has a threshold equilibrium either of the HH/LL or HL/LH type. Then: (i) if the best workers sort into the labor-intensive sector then an increase in $\bar{H}/\bar{L}$ raises the cutoff q_L^* and if the best workers sort into the manager-intensive sector then an increase in $\bar{H}/\bar{L}$ reduces the cutoff q_L^*; and (ii) if the best managers sort into the labor-intensive sector, then an increase in $\bar{H}/\bar{L}$ raises the cutoff q_H^* and if the best managers sort into the manager-intensive sector then an increase in $\bar{H}/\bar{L}$ reduces the cutoff q_H^*.*

To prove this lemma, label the sectors so that the best workers sort into sector 1. We first prove the result for an HH/LL equilibrium and then for an HL/LH equilibrium.

HH/LL Equilibrium

In an HH/LL equilibrium the cutoffs $\{q_H^*, q_L^*\}$ satisfy:

$$w_1(q_L^*) = w_2(q_L^*), \tag{43}$$

$$r_1(q_H^*) = r_2(q_H^*), \tag{44}$$

where $[w_i(\cdot), r_i(\cdot), m_i(\cdot)]$ is a solution to the single-sector differential equations (36) and (41) for $i = 1, 2$ with the boundary conditions

$$m_2(q_{H\min}) = q_{L\min}, \quad m_2(q_H^*) = q_L^*, \tag{45}$$

$$m_1(q_H^*) = q_L^*, \quad m_1(q_{H\max}) = q_{L\max}. \tag{46}$$

Clearly, the solutions for the wage function, the salary function, and the matching functions depend on the parameters of the model, such as prices and factor endowments, as do the equilibrium cutoffs $\{q_H^*, q_L^*\}$. We denote by $dw_i(q_L)/d\vartheta$ the derivative of the wage function in sector i with respect to a parameter ϑ, where this derivative accounts for the endogenous adjustments of all three functions. This derivative contrasts with $w_i'(q_L)$, which is the slope of the wage function for given parameters. We use similar notation to represent derivatives of the salary function.

For now, we are interested in $\eta = \bar{H}/\bar{L}$ and we shall use the following elasticities

$$\varepsilon_{w_i,\eta}^* = \left.\frac{dw_i(q_L)}{d\left(\bar{H}/\bar{L}\right)} \cdot \frac{\bar{H}/\bar{L}}{q_L}\right|_{q_L=q_L^*}, \quad \varepsilon_{r_i,\eta}^* = \left.\frac{dr_i(q_H)}{d\left(\bar{H}/\bar{L}\right)} \cdot \frac{\bar{H}/\bar{L}}{q_H}\right|_{q_H=q_H^*}.$$

Differentiating (43)-(44) with respect to $\eta \equiv \bar{H}/\bar{L}$ yields

$$\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] dq_L^* = \varepsilon_{w_2,\eta}^* - \varepsilon_{w_1,\eta}^*, \tag{47}$$

$$\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] dq_H^* = \varepsilon_{r_2,\eta}^* - \varepsilon_{r_1,\eta}^*. \tag{48}$$

The assumptions that the equilibrium is of the HH/LL type and that the best workers and managers sort into sector 1 imply that the expressions in the square brackets are positive in both equations; that is, at the boundary $\{q_H^*, q_L^*\}$ between the two sectors the slopes of the wage and salary functions have to be steeper in sector 1 into which the more able employees sort. It follows that q_L^* rises in response to an increase in the ratio of managers to workers if and only if $\varepsilon_{w_2,\eta}^* > \varepsilon_{w_1,\eta}^*$ and the cutoff q_H^* rises if and only if $\varepsilon_{r_2,\eta}^* > \varepsilon_{r_1,\eta}^*$.

To understand the elasticities $\varepsilon_{w_i,\eta}^*$ and $\varepsilon_{r_i,\eta}^*$, note that a shift in $\bar{H}/\bar{L}$ impacts wages and salaries through two channels. First, there is the direct effect described in part (iii) of Lemma 1, which adds $1-\gamma_i$ to $\varepsilon_{w_i,\eta}^*$ and $-\gamma_i$ to $\varepsilon_{r_i,\eta}^*$. This stems from the fact that, with constant boundaries, factor endowments do not affect the matching functions. But given factor intensity differences across sectors, equations (47) and (48) imply that with no changes in matching the right-hand side of each one of these equations equals $\gamma_1 - \gamma_2$, which generate an increase in q_L^* and q_H^* if and only if $\gamma_1 - \gamma_2 > 0$. These shifts in the cutoffs trigger re-matching in each sector, which impacts in turn the wage and salary functions, as implied by Lemmas 3-6 and Corollaries 1 and 2 to Lemma 6. In other words, the impact effect of a rise in $\bar{H}/\bar{L}$ increases the cutoffs for both workers and managers, but we also have to account for the induced change in matching in order to obtain the full effect. To this end, we now express the elasticities $\varepsilon_{w_i,\eta}^*$ and $\varepsilon_{r_i,\eta}^*$ as follows:

$$\varepsilon_{w_i,\eta}^* = (1-\gamma_i)\,\hat{\eta} + \varepsilon_{w_iL}^*\hat{q}_L^* + \varepsilon_{w_iH}^*\hat{q}_H^*, \quad i = 1,2, \tag{49}$$

$$\varepsilon_{r_i,\eta}^* = -\gamma_i\hat{\eta} + \varepsilon_{r_iL}^*\hat{q}_L^* + \varepsilon_{r_iH}^*\hat{q}_H^*, \quad i = 1,2, \tag{50}$$

where $1-\gamma_i$ and $-\gamma_i$ represent the direct impacts of $\bar{H}/\bar{L}$, $\varepsilon_{w_iL}^*$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_L^* through the induced re-matching (evaluated at q_L^*), and $\varepsilon_{w_iH}^*$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_H^* through the induced re-matching (evaluated at q_L^*). From (35) and (36) we also have

$$\varepsilon_{r_iF}^* = -\frac{\gamma_i}{1-\gamma_i}\varepsilon_{w_iF}^*, \quad F = H, L; \quad i = 1,2. \tag{51}$$

Now substitute these equations into (47) and (48) to obtain

$$M_h^{HH/LL}\begin{pmatrix} \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} \gamma_1 - \gamma_2 \\ \gamma_1 - \gamma_2 \end{pmatrix}\hat{\eta}, \tag{52}$$

where

$$M_h^{HH/LL} = \begin{pmatrix} q_L^*\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] + \varepsilon_{w_1L}^* - \varepsilon_{w_2L}^* & \varepsilon_{w_1H}^* - \varepsilon_{w_2H}^* \\ \frac{\gamma_2\varepsilon_{w_2L}^*}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1L}^*}{1-\gamma_1} & q_H^*\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] + \frac{\gamma_2\varepsilon_{w_2H}^*}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1H}^*}{1-\gamma_1} \end{pmatrix}.$$

From Lemmas 3-6 we have

$$\varepsilon_{w_1L}^* > 0, \quad \varepsilon_{w_2L}^* < 0, \quad \varepsilon_{w_1H}^* < 0, \quad \varepsilon_{w_2H}^* > 0.$$

These equations provide a solution to $\hat{q}_L^*$ and $\hat{q}_H^*$.

The determinant of the matrix $M_h^{HH/LL}$ is

$$D_{M_h^{HH/LL}} = \left\{ q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)} \right] + \varepsilon_{w_1 L}^* - \varepsilon_{w_2 L}^* \right\} q_H^* \left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)} \right]$$
$$+ \left(\frac{\gamma_2 \varepsilon_{w_2 H}^*}{1-\gamma_2} - \frac{\gamma_1 \varepsilon_{w_1 H}^*}{1-\gamma_1} \right) q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)} \right] - \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)} \left(\varepsilon_{w_2 H}^* \varepsilon_{w_1 L}^* - \varepsilon_{w_1 H}^* \varepsilon_{w_2 L}^* \right).$$

The first two terms on the right-hand side are positive. We now show that the third term also is positive. To this end, note from Lemma 2 that if we change a single boundary and the new boundary is on the original matching function then the new matching function coincides with the old one in the overlapping range of abilities. Therefore, if we choose $dq_L^* = m_i'(q_H^*)\, dq_H^*$, where $m_i(\cdot)$ is the solution of matching in sector i, then a change in the boundary (dq_H^*, dq_L^*) does not change the wage $w_i(q_L^*)$. In other words,

$$\varepsilon_{w_i H}^* + \varepsilon_{w_i L}^* \varepsilon_{m_i}^* = 0,$$

where $\varepsilon_{m_i}^*$ is the elasticity of $m_i(\cdot)$ evaluated at q_H^*. On the other hand, (37) implies for the HH/LL case that

$$\varepsilon_{m_i}^* = \frac{\kappa_m \gamma_i}{1-\gamma_i},$$

where

$$\kappa_m = \frac{\bar{H} r(q_H^*)\, \phi_H(q_H^*)\, q_H^*}{\bar{L} w(q_L^*)\, \phi_L(q_L^*)\, q_L^*}.$$

Therefore,

$$\varepsilon_{w_i H}^* = -\frac{\kappa_m \gamma_i}{1-\gamma_i} \varepsilon_{w_i L}^*.$$

Using this expression, we obtain

$$-\frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)} \left(\varepsilon_{w_2 H}^* \varepsilon_{w_1 L}^* - \varepsilon_{w_1 H}^* \varepsilon_{w_2 L}^* \right) = -\frac{(\gamma_1-\gamma_2)^2 \kappa_m \varepsilon_{w_1 L}^* \varepsilon_{w_2 L}^*}{(1-\gamma_1)^2 (1-\gamma_2)^2} > 0,$$

which proves that $D_{M_h^{HH/LL}} > 0$.

Solving (52) implies that $\hat{q}_L^* > 0$ and $\hat{q}_H^* > 0$ if and only if $(\gamma_1 - \gamma_2)\hat{\eta} > 0$. In other words, a rise in $\bar{H}/\bar{L}$ increases both cutoffs if and only if sector 1 is labor intensive.

Next consider the effects of price changes. An increase in the price of good i raises on impact wages and salaries in sector i by $\hat{p}_i$ and has no direct impact on wages and salaries in the other sector. Following the previous arguments, the change in the equilibrium cutoff points can be found as the solution to

$$M_h^{HH/LL} \begin{pmatrix} \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} \hat{p}_2 - \hat{p}_1 \\ \hat{p}_2 - \hat{p}_1 \end{pmatrix}, \tag{53}$$

where the matrix $M_h^{HH/LL}$ is the same as in (52). It follows from this system that $\hat{q}_L^* > 0$ and $\hat{q}_H^* > 0$ if and only if $\hat{p}_2 > \hat{p}_1$. That is, an increase in the relative price of good 2 raises both cutoffs and therefore raises output in sector 2 and reduces that in sector 1.

HL/LH Equilibrium

In an HL/LH equilibrium, the cutoffs $\{q_H^*, q_L^*\}$ also satisfy the continuity conditions (43) and (44), but the boundary conditions are different. Assuming as before that the best workers sort into sector 1, this means that in an HL/LH equilibrium the best managers sort into sector 2 and the boundary conditions are

$$m_1(q_{H\min}) = q_L^*, \quad m_1(q_H^*) = q_{L\max},$$

$$m_2(q_H^*) = q_{L\min}, \quad m_2(q_{H\max}) = q_L^*.$$

Figure 5 depicts the pattern of sorting and matching in this type of equilibrium. The more-able workers sort into sector 1 only if

$$\frac{w_1'(q_L^*)}{w_1(q_L^*)} > \frac{w_2'(q_L^*)}{w_2(q_L^*)}$$

and the more-able managers sort into sector 2 only if

$$\frac{r_1'(q_H^*)}{r_1(q_H^*)} < \frac{r_2'(q_H^*)}{r_2(q_H^*)}.$$

To derive the comparative statics, we use as before conditions (47) and (48), which apply in this case too. We also can use the decomposition of elasticities (49) and (50), which still apply. Now, however, the relationship between the elasticities of the salary and wage functions, as described by (51), does not apply, because workers of ability q_L^* do not pair with managers of ability q_H^*, as is evident from Figure 5. Instead, from (35) and (36) we now obtain

$$\varepsilon_{r_1F}^* = -\frac{\gamma_1}{1-\gamma_1}\varepsilon_{w_1F}^{\max}, \quad F = H, L,$$

$$\varepsilon_{r_2F}^* = -\frac{\gamma_2}{1-\gamma_2}\varepsilon_{w_2F}^{\min}, \quad F = H, L,$$

where $\varepsilon_{r_iF}^*$ is defined in the same way as before, $\varepsilon_{w_1F}^{\max}$ is the elasticity of $w_1(\cdot)$ with respect to the boundary q_F^* through the induced re-matching in sector 1 (evaluated at $q_{L\max}$) and $\varepsilon_{w_2F}^{\min}$ is the elasticity of $w_2(\cdot)$ with respect to the boundary q_F^* through the induced re-matching in sector 2 (evaluated at $q_{L\min}$). Using these results the systems of equations (52) and (53) are replaced by

$$M_h^{HL/LH}\begin{pmatrix}\hat{q}_L^*\\ \hat{q}_H^*\end{pmatrix} = \begin{pmatrix}\gamma_1-\gamma_2\\ \gamma_1-\gamma_2\end{pmatrix}\hat{\eta}, \tag{54}$$

and

$$M_h^{HL/LH}\begin{pmatrix}\hat{q}_L^*\\ \hat{q}_H^*\end{pmatrix} = \begin{pmatrix}\hat{p}_2-\hat{p}_1\\ \hat{p}_2-\hat{p}_1\end{pmatrix}, \tag{55}$$

where

$$M_h^{HL/LH} = \begin{pmatrix} q_L^*\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] + \varepsilon_{w_1L}^* - \varepsilon_{w_2L}^* & \varepsilon_{w_1H}^* - \varepsilon_{w_2H}^* \\ \frac{\gamma_2\varepsilon_{w_2L}^{\min}}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1L}^{\max}}{1-\gamma_1} & q_H^*\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] + \frac{\gamma_2\varepsilon_{w_2H}^{\min}}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1H}^{\max}}{1-\gamma_1} \end{pmatrix}. \tag{56}$$

From Lemmas 3-6, we have $\varepsilon_{w_1L}^* > 0 > \varepsilon_{w_2L}^*$, $\varepsilon_{w_1H}^* > 0 > \varepsilon_{w_2H}^*$, $\varepsilon_{r_1H}^* < 0 < \varepsilon_{r_2H}^*$, $\varepsilon_{r_1L}^* < 0 < \varepsilon_{r_2L}^*$. This implies that both entries in the top row in (56) are strictly positive and both entries in the bottom row are strictly negative.

Consider (55). The previous observations imply that a positive term $\hat{p}_2 - \hat{p}_1$ either raises q_L^* and reduces q_H^*, or it reduces q_L^* and raises q_H^*. The cutoffs cannot both move in the same direction, because the effect in the top row on the left hand side of (55) would then be opposite to those in the bottom row, whereas on the right hand side both effects have the same sign. We will show that only a rise in q_L^* and a reduction q_H^* can be associated with equilibrium responses, which implies that the determinant of $M_h^{HL/LH}$ must be negative ($D_{M_h^{HL/LH}} < 0$). To prove this, consider an increase in the price p_2 to $p_2' > p_2$ while the price p_1 stays constant. Let X_1 and X_2 denote the output in each sector prior to the price change, and let X_1' and X_2' denote the corresponding output after the price change. Since only prices have changed (and not endowments), under each set of prices both the outputs (X_1, X_2) and (X_1', X_2') are feasible. Since the competitive equilibrium is efficient, the value of output is maximized given prices, which implies that

$$\begin{aligned} p_1 X_1 + p_2 X_2 &\geq p_1 X_1' + p_2 X_2', \\ p_1 X_1 + p_2' X_2 &\leq p_1 X_1' + p_2' X_2', \end{aligned}$$

where the first inequality states that prior to the price change the value of output is higher under production bundle (X_1, X_2) than under (X_1', X_2'), while the opposite holds after the price change. Subtracting and rearranging gives

$$(p_2 - p_2')(X_2 - X_2') \geq 0,$$

which implies that $X_2 \leq X_2'$. An increase in output in sector two cannot be achieved with a fall in q_L^* and a rise q_H^*, because in this case there would be less worker types and less manager types in sector 2. Therefore, an increase in the relative price of good 2 leads to a rise in q_L^* and a reduction q_H^*. This requires $D_{M_h^{HL/LH}} < 0$.

Now consider system (54). Since $D_{M_h^{HL/LH}} < 0$, a rise in the relative endowment $\eta \equiv \bar{H}/\bar{L}$ of managers raises q_L^* and reduces q_H^*. Finally, we must determine the effect of a change in the relative endowment of managers on relative outputs, which is affected both by re-matching and the change in endowments. Sector i pays managers a fraction $1 - \gamma_i$ of revenue. Therefore, in an HL/LH equilibrium, we have

$$(1 - \gamma_1)\, p_1 X_1 = \bar{H} \int_{q_{H\min}}^{q_H^*} r(q_H)\, \phi_H(q_H)\, dq_H,$$

$$(1 - \gamma_2)\, p_2 X_2 \equiv \bar{H} \int_{q_H^*}^{q_{H\max}} r(q_H)\, \phi_H(q_H)\, dq_H,$$

which implies

$$\frac{(1 - \gamma_2)\, p_2 X_2}{(1 - \gamma_1)\, p_1 X_1} = \frac{\int_{q_H^*}^{q_{H\max}} r(q_H)\, \phi_H(q_H)\, dq_H}{\int_{q_{H\min}}^{q_H^*} r(q_H)\, \phi_H(q_H)\, dq_H}.$$

From (18) we obtain

$$\ln r_i(q_H) - \ln r_i(q_{H0}) = \int_{q_{H0}}^{q_H} \frac{\psi_{iH}[x, m(x)]}{(1 - \gamma_i)\, \psi_i[x, m(x)]} dx, \text{ for all } q_H, q_{H0} \in Q_{Hi}.$$

Substituting this equation into the previous one yields

$$\frac{X_2}{X_1} = \frac{(1-\gamma_1)\, p_1 \int_{q_H^*}^{q_{H\max}} \exp\left[\int_{q_H^*}^{q_H} \frac{\psi_{2H}[q,m(q)]}{(1-\gamma_2)\psi_2[q,m(q)]} dq\right] \phi_H(q_H)\, dq_H}{(1-\gamma_2)\, p_2 \int_{q_{H\min}}^{q_H^*} \exp\left[-\int_{q_H}^{q_H^*} \frac{\psi_{1H}[q,m(q)]}{(1-\gamma_1)\psi_1[q,m(q)]} dq\right] \phi_H(q_H)\, dq_H}, \tag{57}$$

where we have used the property that $r(\cdot)$ is a continuous function. When sector 2 is manager intensive, q_H^* is lower in country A, which has more managers per worker. We have shown above that, in such circumstances, managers of a given type are teamed with higher-ability workers in country A. Due to the strict log supermodularity of the productivity functions this implies that $\psi_{iH}[q,m(q)]/\psi_i[q,m(q)]$ is higher in country A in both sectors. It follows that the impact of a higher $\bar{H}/\bar{L}$ on matching raises the relative output of good 2. In the opposite case, when $\gamma_1 < \gamma_2$, the shift in matching reduces the relative output of good 2. In short, the shift in matching raises the relative output of the manager-intensive good.

To complete the analysis of the impact of factor endowments on relative outputs, we need to assess the direct impact of the cutoff q_H^* on the relative outputs in (57). First note that q_H^* affects relative outputs through the boundaries of four integrals. When $\gamma_1 > \gamma_2$ and q_H^* declines in response to an increase in $\bar{H}/\bar{L}$, the shifts in the boundaries of the outer integrals in the numerator and denominator raise the relative output of good 2. In the opposite case, when $\gamma_1 < \gamma_2$ and q_H^* rises, the relative output of good 2 declines. A shift in the boundaries of the two inner integrals in the numerator and denominator have opposite effects from one another. Consequently, we need to evaluate their relative strength. Differentiation with respect to these boundaries yields:

$$-\frac{X_2}{X_1}\left\{\lim_{q\searrow q_H^*} \frac{\psi_{2H}[q,m(q)]}{(1-\gamma_2)\,\psi_2[q,m(q)]} - \lim_{q\nearrow q_H^*} \frac{\psi_{1H}[q,m(q)]}{(1-\gamma_1)\,\psi_1[q,m(q)]}\right\}.$$

Since the best managers sort into sector 2, this requires the slope of the salary function $r(\cdot)$ to be steeper at q_H^* in sector 2, or, using (18), it requires the term in the curly bracket to be positive. It follows that a decline in q_H^* raises the relative output of good 2. If instead good 2 is labor intensive, q_H^* rises in response to an increase in $\bar{H}/\bar{L}$, which raises the relative output of good 1. In either case, country A produces relatively more of the manager-intensive good.

Appendix B

In this appendix, we analyze the limiting case of Cobb-Douglas productivity; that is

$$\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i} \text{ for } i = 1, 2;\ \alpha_i, \beta_i > 0. \tag{58}$$

Note that, in this case, productivity is a weakly log supermodular function of the two ability levels. As such, the complementarity between the talent of workers and that of the manager is somewhat muted compared to what arises with strict log supermodularity, which means that the forces for positive assortative matching within a sector are correspondingly weaker.

There is no need to go through all the steps of a firm's profit maximization problem, because the derivation proceeds much as for the case with homogeneous managers in Section 3. Suffice it to say that the demand per manager for workers of ability q_L by a firm in industry i that pairs these workers with a manager of ability q_H is given by

$$\ell(q_L, q_H) = \left[\frac{\gamma_i p_i q_H^{\beta_i} q_L^{\alpha_i}}{w(q_L)}\right]^{\frac{1}{1-\gamma_i}}. \tag{59}$$

Substituting (59) into the expression for profits yields

$$\tilde{\pi}_i(q_L, q_H) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r(q_H), \tag{60}$$

where $r(q_H)$ is the salary of a manager with ability q_H and $\bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}}(1-\gamma_i)$. Every firm chooses the ability of its workers and the ability of its manager so as to maximize profits, yet free entry dictates that these profits must be equal to zero in equilibrium. Let M_i be the set of all matches that maximize profits in sector i. For each pairing (q_L, q_H) in M_i,

$$r(q_H) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}},\quad i = 1, 2, \tag{61}$$

by dint of the zero-profit condition. Profit maximization with respect to the choice of types, evaluated for pairings that achieve zero profits in accordance with (61), yields the first-order conditions,

$$\frac{\alpha_i}{\gamma_i} = \varepsilon_w(q_L) \quad \text{for } q_L \in Q_{Li}^{int} \tag{62}$$

and

$$\frac{\beta_i}{1-\gamma_i} = \varepsilon_r(q_H) \quad \text{for } q_H \in Q_{Hi}^{int}. \tag{63}$$

Equation (62) is the analog to (4) and equates the ratio of the elasticities of output with respect to worker ability and labor quantity to the elasticity of the wage schedule. Equation (63) has a similar interpretation regarding a firm's choice of manager type.

In equilibrium, all worker types must be employed, which means that firms in some sector (or both) must demand the full range of workers. Equation (62) can be satisfied for a range of workers only if the wage schedule has a constant elasticity over this range. Therefore, the equilibrium wage schedule must take the form

$$w(q_L) = w_i q_L^{\alpha_i/\gamma_i} \quad \text{for } q_L \in Q_{Li}^{int}. \tag{64}$$

The salary schedule for managers must have a similar form, namely

$$r(q_H) = r_i q_H^{\beta_i/(1-\gamma_i)} \quad \text{for } q_H \in Q_{Hi}^{int}, \tag{65}$$

where r_i is a "salary anchor" analogous to w_i.

When the wage function has a constant elasticity equal to α_i/γ_i for a range of worker types, a firm in sector i is indifferent as to its choice of employees among workers in this range, irrespective of the ability of its manager. And when the salary function has an elasticity equal to $\beta_i/(1-\gamma_i)$, the firm is indifferent to the ability of its managers. Accordingly, the matching of workers and managers among those that sort to sector i is indeterminate in the Cobb-Douglas case. This indeterminacy reflects the fact that the productivity function in (58) is only weakly log supermodular and thus provides no clear incentives for positive (or negative) assortative matching.

Although the matching of workers and managers in a sector is not determined in the Cobb-Douglas case, the sorting of these factors to the two sectors follows a familiar pattern. The elasticity of the wage schedule must be greater along its upper segment than along its lower segment, or else firms that hire the less able workers would all prefer to upgrade their employees. Similarly, the elasticity of the salary schedule must be greater along its upper segment than its lower segment. We designate as sector 1 whichever industry has the greater ratio of the output elasticity with respect to worker ability to the output elasticity with respect to labor quantity. With this labeling convention, $s_L = \alpha_1/\gamma_1 - \alpha_2/\gamma_2 > 0$. Then, in any equilibrium in which a country produces both goods, sector 1 attracts the workers with ability q_L above some cutoff q_L^*. If $s_H = \beta_1/(1-\gamma_1) - \beta_2/(1-\gamma_2) > 0$, then sector 1 also attracts the more able managers with $q_H > q_H^*$; otherwise, the sorting of managers is opposite to that for workers.

For precision, we state more formally the environment we consider throughout this appendix and the sorting pattern that results.[41]

Assumption 3 (i) $S_H = [q_{H\min}, q_{H\max}]$, $0 < q_{H\min} < q_{H\max} < +\infty$; (ii) $\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i}$, $\alpha_i, \beta_i > 0$, for $i = 1, 2$; and (iii) $s_L \equiv \alpha_1/\gamma_1 - \alpha_2/\gamma_2 > 0$.

Proposition 10 *Suppose that Assumption 3 holds. Then, in any competitive equilibrium with employment in both sectors, the more able workers with $q_L \geq q_L^*$ are employed in sector 1 and the less able workers with $q_L \leq q_L^*$ are employed in sector 2, for some $q_L^* \in S_L$. If $s_H > 0$ ($s_H < 0$), the more able managers with $q_H \geq q_H^*$ are employed in sector 1 (sector 2) and the less able managers with $q_H \leq q_H^*$ are employed in sector 2 (sector 1), for some $q_H^* \in S_H$.*

To describe the equilibrium once the sorting pattern has been settled, we invoke factor-market clearing, continuity of worker wages, continuity of managerial salaries, and the zero-profit conditions. For concreteness,

[41] Proofs of all Propositions stated in this appendix are provided at the end.

let us focus on the case in which $s_H > 0$ so that the more able managers sort to industry 1; the opposite case can be handled similarly.

It proves convenient to define $e_{Hi}(q_H) = q_H^{\beta_i/(1-\gamma_i)}$ as the effective managerial input of a manager with ability q_H who works in sector i. Then the aggregate supplies of effective managerial input in sectors 1 and 2 are

$$H_1 = \bar{H} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H, \tag{66}$$

and

$$H_2 = \bar{H} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H\ , \tag{67}$$

respectively. Note that $H_1/\bar{H}$ depends only on q_H^* and is a monotonically decreasing function, and $H_2/\bar{H}$ also depends only on q_H^* and is monotonically increasing.

Consider now the supply and demand for effective labor in sector 1, where we define $e_{Li}(q_L) = q_L^{\alpha_i/\gamma_i}$ as the effective labor provided by a worker of ability q_L in sector i. From the labor demand equation (59), a firm in sector 1 combines a manager with e_{Hi} units of effective managerial input with $e_{Hi}(\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$ units of effective labor. Therefore, the H_1 units of effective managerial input that are hired into sector 1 are combined with $H_1(\gamma_1 p_1/w_1)^{1/(1-\gamma_1)}$ units of effective labor. Noting the definition of H_1 and equating the demand for effective labor in sector 1 with the supply of effective labor among those with ability above q_L^*, we have

$$\bar{H}\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_L^*}^{q_{L\max}} q_L^{\frac{\alpha_1}{\gamma_1}} \phi_L dq_L\ . \tag{68}$$

A similar condition applies in sector 2, where labor-market clearing requires

$$\bar{H}\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_{L\min}}^{q_L^*} q_L^{\frac{\alpha_2}{\gamma_2}} \phi_L dq_L\ . \tag{69}$$

Continuity of the wage schedule at q_L^* requires that

$$w_1 (q_L^*)^{\frac{\alpha_1}{\gamma_1}} = w_2 (q_L^*)^{\frac{\alpha_2}{\gamma_2}}\ . \tag{70}$$

The salary function for managers must also be continuous and firms that hire managers with ability q_H^* must earn zero profits in either sector. Together, these considerations imply

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} (q_H^*)^{\frac{\beta_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}} (q_H^*)^{\frac{\beta_2}{1-\gamma_2}}\ . \tag{71}$$

Equations (68)-(71) comprise four equations that can be used to solve for the two wage anchors, w_1 and w_2, and the two cutoffs, q_L^* and q_H^*. The effective supply of managers in sectors 1 and 2, H_1 and H_2, can then be solved from (66) and (67). Finally, the salary anchors for the managers can be computed from the zero-profit conditions, which imply

$$r_i = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} w_i^{-\frac{\gamma_i}{1-\gamma_i}} \quad \text{for } i = 1, 2. \tag{72}$$

This completes our characterization of the supply-side equilibrium for an economy that faces prices p_1 and p_2.

Pattern of Trade

As in Section 3, we need an expression for an economy's relative outputs in order to conduct the comparative static analysis that reveals the pattern of trade between countries that differ in their relative factor endowments or in their distributions of factor types. The H_i units of effective managers employed in sector i collectively produce $X_i = H_i (\gamma_i p_i)^{\gamma_i/(1-\gamma_i)} w_i^{-\gamma_i/(1-\gamma_i)}$ units of good i. Each effective unit of managerial input is paid a salary of r_i in sector i and—by continuity of the salary function—$r_1/r_2 = (q_H^*)^{-s_H}$ (see (65)). Using this condition together with (68)-(69) and (71)-(72), we can write

$$\frac{X_1}{X_2} = \frac{r_1 H_1}{r_2 H_2} \frac{(1-\gamma_2)\, p_2}{(1-\gamma_1)\, p_1} = \frac{(1-\gamma_2)\, p_2 \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H}{(1-\gamma_1)\, p_1 \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H} (q_H^*)^{-s_H}. \tag{73}$$

Similar to the case of homogeneous managers, the first equality reflects the fact that the aggregate salaries of all managers in sector i absorb a fraction $1-\gamma_i$ of revenue. And the second equality implies that, since $s_H > 0$ in the case under consideration, X_1/X_2 is a decreasing function of q_H^*. Therefore, to identify the pattern of trade, we need only find which country allocates more effective managerial input to sector 1 relative to its aggregate endowment of managers; that is, how q_H^* varies with factor endowments.[42]

The system of equations (68)-(71) that applies with Cobb-Douglas productivity is quite similar to the system (6)-(9) that applies when managers are homogeneous, except that now we need to use the effective managerial input in a sector in place of the pure number of managers. In other words, the multiplicative separability of the productivity function allows us to construct an aggregate measure of managerial input that plays the same role as does the number of managers when managers are equally productive. We can do so, because there are no forces present in the Cobb-Douglas case to induce any particular pattern of matching within either sector. The following propositions assert that the determinants of the trade pattern in an economy with heterogeneous managers but Cobb-Douglas productivity mirror those that we described for an economy with homogeneous managers.

Proposition 11 *Suppose that Assumption 3 holds. Then if $\phi_L^A(q_L) = \phi_L^B(q_L)$ for all $q_L \in S_L^A = S_L^B$, $\phi_H^A(q_H) = \phi_H^B(q_H)$ for all $q_H \in S_H^A = S_H^B$, and $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$, country A exports the manager-intensive good.*

Proposition 12 *Suppose that Assumption 3 holds and $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$. Then, (i) if $\phi_H^A(q_H) = \phi_H^B(q_H)$ for all $q_H \in S_H^A = S_H^B$ and $\phi_L^A(q_L)$ is a rightward shift of $\phi_L^B(q_L)$ for some $\lambda > 1$, then country A exports good 1 if and only if $\alpha_1 > \alpha_2$; (ii) if $\phi_L^A(q_L) = \phi_L^B(q_L)$ for all $q_L \in S_L^A = S_L^B$ and $\phi_H^A(q_H)$ is a rightward shift of $\phi_H^B(q_H)$ for some $\lambda > 1$, then country A exports good 1 if and only if $\beta_1 > \beta_2$.*

In short, the Heckscher-Ohlin theorem applies when countries have similar distributions of factor types but differ in their relative aggregate endowments of managers versus workers. Alternatively, if the relative factor

[42] Note that in the opposite case, when $s_H < 0$, managers with $q_H \geq q_H^*$ sort into sector 2 while managers with $q_H \leq q_H^*$ sort into sector 1. As a result, X_1/X_2 is an increasing function of q_H^*.

endowments are the same in the two countries but they differ in their distributions of one of the factors, then the country with the rightward-shifted distribution of a factor exports the good produced by the industry in which productivity responds more elastically to that factor's ability.

Effects of Trade on Income Distribution

Our results on income distribution also carry over straightforwardly from the case with homogeneous managers to that with manager heterogeneity but Cobb-Douglas productivity. First note that within-industry income distribution is not affected by world trade inasmuch as the elasticity of the wage schedule for workers employed in a given industry is constant. As a result, (64) implies that $w(q'_L)/w(q''_L) = (q'_L/q''_L)^{\alpha_i/\gamma_i}$ for $q'_L, q''_L \in Q_{Li}$ and (65) implies that $r(q'_H)/r(q''_H) = (q'_H/q''_H)^{\beta_i/(1-\gamma_i)}$ for $q'_H, q''_H \in Q_{Hi}$. Second, relative rewards of workers and managers that are employed in different industries do change with trade, inasmuch as the wage and salary anchors w_i and r_i change. Our next proposition is

Proposition 13 *Suppose that Assumption 3 holds and $s_H \approx 0$. When $\hat{p}_1 > 0$, (i) $\hat{w}_1 > \hat{w}_2$; (ii) if $\gamma_1 \approx \gamma_2$, then $\hat{w}_1 > \hat{p}_1 > \hat{r}_1 \approx \hat{r}_2 > 0 > \hat{w}_2$; (iii) if $\gamma_1 > \gamma_2$ and $s_L \approx 0$, then $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}_1 \approx \hat{r}_2$; (iv) if $\gamma_1 < \gamma_2$ and $s_L \approx 0$, then $\hat{r}_1 \approx \hat{r}_2 > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$.*

Proposition 13 can be understood by recognizing that the model with heterogeneous workers and managers also contains a blend of Stolper-Samuelson and Ricardo-Viner forces. When $s_H \approx 0$, there is no difference in the suitability of the various managers for employment in one sector versus the other, because the comparative advantage associated with greater ability of the input just offsets the comparative advantage associated with greater quantity. Then, it is as if managers are a perfectly mobile, homogeneous factor. When s_L also is small, the Stolper-Samuelson forces will dominate, and workers in both industries will see a gain in real income if the relative price of the labor-intensive good rises and will see a loss in real income if the relative price of the labor-intensive good falls. In contrast, if factor intensities are approximately the same in the two industries, the Stolper-Samuelson forces will be muted, and the partial specificity of workers arising from the comparative advantage of ability in sector 1 will govern the income responses. Then, workers will benefit in real terms when the relative price of the good they produce rises and will lose in real terms if the relative price of this good falls. Also note that similar considerations imply that if $s_H > 0$ but $s_L \approx 0$ and $\gamma_1 \approx \gamma_2$, the economy behaves like one with sector-specific managers and perfectly mobile labor. Then $\hat{r}_1 > \hat{p}_1 > \hat{w}_1 \approx \hat{w}_2 > 0 > \hat{r}_2$, i.e., managers in the expanding sector gain, managers in the contracting sector lose, and workers may gain or lose in real terms depending on their consumption pattern. Finally, similarly to Proposition 4, an increase in the price of good 1 raises overall wage inequality, because it does not change relative wages within sectors and it increases wages of the more able, better-paid workers employed in sector 1 relative to the less able, lower-paid workers in sector 2.

Proofs

First note that, in the system comprising (68)-(71), a proportional increase in the number of managers and workers has no effect on the wage anchors w_1 and w_2 or on the ability cutoffs q_L^* and q_H^*. Therefore, it does not change the output ratio X_1/X_2 (see (73)). It follows that if countries A and B differ only in size, with $\bar{H}$ and $\bar{L}$ being proportionately larger in one of the countries, they will have the same relative demand for the two goods and the same relative supply and they will not trade with one another. Accordingly, we can find the impact of $\bar{H}/\bar{L}$ on the pattern of trade by analyzing the impact of $\bar{L}$ on q_H^*, which will tell us how the relative supply X_1/X_2 is affected.

Differentiating the equilibrium system (68)-(71), we obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_2 \\ -E_1 \end{pmatrix} \hat{\bar{L}} + \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix} \hat{p}_1,$$

where E_i is effective labor in sector i, defined as

$$E_1 = \bar{H} \left(\frac{\gamma_1 p_1}{w_1} \right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H,$$

$$E_2 = \bar{H} \left(\frac{\gamma_2 p_2}{w_2} \right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H,$$

and

$$\Lambda_1 = \bar{L} \left(q_L^*\right)^{\frac{\alpha_1}{\gamma_1}+1} \phi_L\left(q_L^*\right),$$

$$\Lambda_2 = \bar{L} \left(q_L^*\right)^{\frac{\alpha_2}{\gamma_2}+1} \phi_L\left(q_L^*\right),$$

$$\Theta_1 = \bar{H} \left(\frac{\gamma_1 p_1}{w_1} \right)^{\frac{1}{1-\gamma_1}} \left(q_H^*\right)^{\frac{\beta_1}{1-\gamma_1}+1} \phi_H\left(q_H^*\right),$$

$$\Theta_2 = \bar{H} \left(\frac{\gamma_2 p_2}{w_2} \right)^{\frac{1}{1-\gamma_2}} \left(q_H^*\right)^{\frac{\beta_2}{1-\gamma_2}+1} \phi_H\left(q_H^*\right).$$

The determinant of the matrix on the left-hand side of this system, D_{CD}, satisfies

$$\begin{aligned} (1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1 - \gamma_2) + s_H\left[\Lambda_1 E_2 (1-\gamma_1) + \Lambda_2 E_1 (1-\gamma_2)\right] \\ &\quad + s_L\left(\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1\right) + E_1 E_2 s_H s_L. \end{aligned}$$

Using the equilibrium conditions (70) and (71), we find that

$$(\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1 - \gamma_2) = \Theta_2\Lambda_1 \frac{(\gamma_1-\gamma_2)^2}{\gamma_2(1-\gamma_1)} > 0.$$

Therefore $D_{CD} < 0$. We also compute

$$\hat{q}_H^* D_{CD} = (\Lambda_1 E_2 + \Lambda_2 E_1 + E_1 E_2 s_L) \frac{\gamma_1 - \gamma_2}{(1-\gamma_1)(1-\gamma_2)} \hat{\bar{L}}.$$

Since $D_{CD} < 0$, an increase in $\bar{L}$ reduces q_H^* if and only if $\gamma_1 > \gamma_2$. So, the output of good 1 rises relative to that of good 2 if and only if sector 2 is more labor intensive than sector 1. This proves Proposition 11.

Next, we calculate the impact of p_1 on the wage anchors:

$$\begin{aligned}\hat{w}_1(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1 + \Lambda_2 E_1 s_H)(1-\gamma_2)\hat{p}_1 \\ &\quad + [(\Theta_1 E_2 + \Theta_2\gamma_2 E_1) s_L + E_1 E_2 s_H s_L]\hat{p}_1, \\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1 + \Lambda_2 E_1 s_H - \Theta_2 E_1 s_L)\hat{p}_1.\end{aligned}$$

Therefore,

$$(\hat{w}_1 - \hat{w}_2)(1-\gamma_1)(-D_{CD}) = [(\Theta_1 E_2 + \Theta_2\gamma_2 E_1) s_L + E_1 E_2 s_H s_L + \Theta_2 E_1 s_L (1-\gamma_2)]\hat{p}_1.$$

Since $D_{CD} < 0$, it follows that an increase in the price of good 1 results in $\hat{w}_1 > \hat{w}_2$, which proves part (i) of Proposition 13.

Next consider the case in which $s_H \approx 0$ and $\gamma_1 \approx \gamma_2$. In this case,

$$(1-\gamma_2)(1-\gamma_1)(-D_{CD}) \approx s_L(\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1).$$

Then

$$\hat{w}_1 \approx \hat{w}_2 \approx \frac{\Theta_1 E_2 + \Theta_2\gamma_2 E_1}{\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1}\hat{p}_1,$$

because $\gamma_1 \approx \gamma_2$ implies $\Theta_1\Lambda_2 - \Theta_2\Lambda_1 \approx 0$. Evidently, in this case, $\hat{w}_1 > \hat{p}_1 > 0 > \hat{w}_2$. To complete the proof of part (ii) of Proposition 13, we need to calculate the response of the anchors r_1 and r_2 for the managers' salaries. When p_1 rises, (72) yields $\hat{r}_1 = (1-\gamma_1)^{-1}\hat{p}_1 - \gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and $\hat{r}_2 = -\gamma_2(1-\gamma_2)^{-1}\hat{w}_2$. In case (ii) of Proposition 13, with $s_H \approx 0$ and $\gamma_1 \approx \gamma_2$, these imply

$$\hat{r}_1 \approx \hat{r}_2 \approx \frac{\Theta_2\gamma_2 E_1}{\Theta_1\gamma_1 E_2 + \Theta_2\gamma_2 E_1}\hat{p}_1.$$

It follows that $\hat{p}_1 > \hat{r}_1 \approx \hat{r}_2 > 0$. So, part (ii) of the proposition is proved.

We turn now to parts (iii) and (iv) of Proposition 13. The antecedents $s_H \approx 0$ and $s_L \approx 0$ imply

$$\begin{aligned}(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1 - \gamma_2), \\ \hat{w}_1(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)\hat{p}_1, \\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)\hat{p}_1.\end{aligned}$$

It follows that

$$\hat{w}_1 \approx \hat{w}_2 \approx \frac{1-\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1,$$

which implies that $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0$ for $\gamma_1 > \gamma_2$ and $\hat{w}_1 \approx \hat{w}_2 < 0 < \hat{p}_1$ for $\gamma_1 < \gamma_2$. Moreover, since $\hat{r}_1 = (1-\gamma_1)^{-1}\hat{p}_1 - \gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and $\hat{r}_2 = -\gamma_2(1-\gamma_2)^{-1}\hat{w}_2$, we have

$$\hat{r}_1 \approx \hat{r}_2 \approx -\frac{\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1.$$

Evidently, in this case, $\hat{r}_1 \approx \hat{r}_2 < 0 < \hat{p}_1$ when $\gamma_1 > \gamma_2$ and $\hat{r}_1 \approx \hat{r}_2 > \hat{p}_1 > 0$ when $\gamma_1 < \gamma_2$. This completes the proof of Proposition 13.

We next consider the impact of a rightward shift of the density function $\phi(q_L)$, as defined in (12).

To perform these comparative statics, we follow the procedure from the previous section; that is, we hold the distribution of types constant but endow a worker of type q_L with λq_L units of ability, and we define $Q_L^* = \lambda q_L^*$. Differentiating the equilibrium system (68)-(71), we now obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{Q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\alpha_2}{\gamma_2} E_2 + \Lambda_2 \\ -\frac{\alpha_1}{\gamma_1} E_1 - \Lambda_1 \end{pmatrix} \widehat{\lambda}.$$

Using (**??**), it follows that for $s_H > 0$ an increase in λ raises the relative output of good 1 if it reduces q_H^*. However, from the above system of equations we obtain:

$$\hat{q}_H^* \left(1-\gamma_2\right)\left(1-\gamma_1\right)\left(-D_{CD}\right) = -\left(\Lambda_1 E_2 + \Lambda_2 E_1 + s_L E_1 E_2\right)\left(\alpha_1 - \alpha_2\right)\hat{\lambda}.$$

Therefore a rightward shift of the density function $\phi\left(q_L\right)$ raises the relative output of good 1 if and only if $\alpha_1 > \alpha_2$. This proves the first part of Proposition 12. The second part is proved similarly.