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# *Testing the TASP: An Experimental Investigation of Learning in Games with Unstable Equilibria*

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# Testing the TASP: An Experimental Investigation of Learning in Games with Unstable Equilibria[1]


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**Abstract**

We report experiments designed to test between Nash equilibria that are stable and unstable under learning. The "TASP" (Time Average of the Shapley Polygon) gives a precise prediction about what happens when there is divergence from equilibrium under fictitious play like learning processes. We use two 4 x 4 games each with a unique mixed Nash equilibrium; one is stable and one is unstable under learning. Both games are versions of Rock-Paper-Scissors with the addition of a fourth strategy, Dumb. Nash equilibrium places a weight of 1/2 on Dumb in both games, but the TASP places no weight on Dumb when the equilibrium is unstable. We also vary the level of monetary payoffs with higher payoffs predicted to increase instability. We find that the high payoff unstable treatment differs from the others. Frequency of Dumb is lower and play is further from Nash than in the other treatments. That is, we find support for the comparative statics prediction of learning theory, although the frequency of Dumb is substantially greater than zero in the unstable treatments.



**JEL numbers:** C72, C73, C92, D83

**Keywords:** games, experiments, TASP, learning, unstable, mixed equilibrium, fictitious play.


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## 1. Introduction

Economic models often only have equilibria in mixed strategies, but it is difficult to see how actual market participants know how to randomize with the correct probabilities. Recent theoretical advances, in particular the development of stochastic fictitious play, demonstrate that, fortunately, even if agents only follow simple learning rules, it is still possible to learn equilibrium behavior. The downside to this is that it has been shown that there are a large number of games for which the equilibria are not learnable. Players following any one of a range of learning processes would not converge to equilibrium. However, Benaïm, Hofbauer and Hopkins (2009) show that, nonetheless, stochastic fictitious play can give a point prediction for play even when it diverges. This point is the TASP (Time Average of the Shapley Polygon) which they show can be quite distinct from any Nash equilibrium.

In this paper, we report experiments designed to test between Nash equilibria that are stable and unstable under learning. Subjects were randomly matched to play one of two 4 x 4 games each with a unique mixed Nash equilibrium. In one game, the equilibrium is predicted to be stable under learning, and in the other unstable. Both games are versions of Rock-Paper-Scissors with the addition of a fourth strategy, Dumb. The mixed equilibrium in both games is (1, 1, 1, 3)/6: Dumb is thus the most frequent strategy. However, in the unstable game, fictitious play-like learning processes are predicted to diverge from the equilibrium to a cycle, a "Shapley polygon," that places no weight upon Dumb. Thus, if fictitious play describes agents' behavior, the limiting frequency of Dumb is a ready indicator of whether we are in the stable or unstable case. It is also, therefore, a simple way to determine whether the predictions of fictitious play, and learning theory, hold in practice. Equilibrium theory suggests that the frequency of Dumb should be the same in both games. Learning theory suggests they should be quite different.

The experiment has a 2 x 2 design with four treatment conditions: unstable or stable game and high or low payoff. This is because both the theory of quantal response equilibria (QRE) and stochastic fictitious play (SFP) predict that multiplying all payoffs by a positive constant will affect play. Specifically, QRE predicts higher payoffs result in play closer to Nash equilibrium, but SFP predicts that higher payoffs will lead to play being further from Nash in the unstable game. We find that there is a difference in play in the high payoff unstable treatment. The frequency of Dumb is lower and play is further from Nash than in the other treatments. That is, we find support for the comparative statics prediction of learning theory even though the frequency of Dumb is substantially greater than zero in the unstable games. The data also reject Nash equilibrium, which predicts no difference between the treatments and QRE theory that predicts play should be closer to Nash in the high payoff treatments.

Fictitious play has the underlying principle that players select a best response to their beliefs about opponents. Traditionally, these beliefs are constructed from the average past play of opponents. This we refer to as players having "classical" beliefs. However, experimental work has found greater success with generalizations of fictitious play that allow for players constructing beliefs by placing greater weight on more recent events (see Cheung and Friedman (1997), Camerer and Ho (1999) amongst many others). This is called forgetting or recency or weighted fictitious play.

In both the existing empirical and theoretical literature on mixed strategy equilibria, there are two principal criteria for determining whether players do actually play a mixed strategy equilibrium. For example, Foster and Young (2003) make the distinction between what they call convergence in *time average* and convergence in *behavior*. The first requires the overall frequencies of play to approach the mixed strategy equilibrium frequencies. The second requires the more demanding standard that players should actually come to randomize with equilibrium probabilities. To illustrate the difference the sequence 0,1,0,1,0,1,... converges in average to 1/2 but clearly not to the behavior of randomizing between 0 and 1 with equal probabilities.

In the experimental literature, this distinction was first raised by Brown and Rosenthal (1990). Indeed, their analysis of the earlier experiments of O'Neill (1987) finds that while play converged in time average, it failed to do so in behavior, in that there was significant autocorrelation in play. Subsequent experiments on games with mixed strategy equilibria seem to confirm this finding. For example, Brown Kruse et al. (1994), Cason and Friedman (2003) and Cason, Friedman and Wagener (2005) find in oligopoly experiments that the average frequencies of prices approximate Nash equilibrium frequencies. However, there are persistent cycles in the prices charged, which seems to reject convergence to equilibrium in behavior.[2]

Returning to theory, even when fictitious play converges to a mixed strategy

[2]A recent exception to this run of results is that Palacios-Huerta and Volij (2008) find that expert players (professional sportsmen) can in fact learn equilibrium behavior. However, Levitt et al. (2007) report additional experiments in which professionals do no better than students.

equilibrium, it does so only in time average not in behavior. This problem motivated the introduction of smooth or stochastic fictitious play (Fudenberg and Kreps (1993)), which permits asymptotic convergence in behavior to mixed strategy equilibria. This more recent work still employs classical beliefs. When one considers weighted stochastic fictitious play, there is not even this asymptotic convergence - play can converge close to equilibrium but never quite lose autocorrelation. This creates a significant problem in research on mixed strategy equilibria in the laboratory. If play is not i.i.d. over the finite length of an experiment, is this because play is diverging, because convergence will never be better than approximate, or because convergence is coming but has not yet arrived?

The current experiment attempts to sidestep these problems by not measuring convergence in terms of the time series properties of play. Rather, the advantage of the game we consider is that a considerable qualitative difference in behavior is predicted between its stable and unstable versions. The result proved in Benaïm, Hofbauer and Hopkins (2009) is that, when players learn according to weighted fictitious play in a class of games in which learning diverges from the mixed equilibrium, the time average of play converges. However, it does not converge to the equilibrium but to the TASP, a new concept. In the unstable game we consider, the TASP is quite distinct from the unique Nash equilibrium. Thus, an easy test of divergence is simply to see whether average play is closer to the TASP or the Nash equilibrium.

In practice, one cannot expect play to be exactly at either the Nash equilibrium or the TASP. The now extensive literature on perturbed equilibria such as quantal response equilibria (QRE) (McKelvey and Palfrey, 1995) makes clear that play in experiments can be quite distinct from Nash equilibrium. Subjects appear to behave as though their choices were subject to noise. Equally, since the stationary points of stochastic fictitious play are QRE, learning theory can make similar predictions. Thus we should expect learning to converge exactly to the TASP only in the absence of noise. In both cases, the theory predicts that the effective level of noise should be sensitive to the level of the payoffs. This type of effect has been found empirically by Battalio et al. (2001) and Bassi et al. (2006). Thus, the other aspect of our design is to change the level of monetary rewards. We ran both the stable and unstable game at two different conversion rates between experimental francs and U.S. dollars, with the high conversion rate two and a half times higher than the lower.

Learning theory predicts that this change in monetary compensation will have a different comparative static effect in the two different games. Higher payoffs should make play diverge further from the equilibrium in the unstable game and make play closer in the stable one. QRE theory, which as an equilibrium theory does not consider stability issues, predicts play should be closer to Nash equilibrium when payoffs are higher, in both the stable and unstable games. Nash equilibrium predicts no difference across the treatments. That is, we have clear and distinct comparative statics predictions to test.

Other experimental studies have tested for differences in behavior around stable and unstable mixed equilibria. Tang (2001) and Engle-Warnick and Hopkins (2006) look at stable and unstable 3 x 3 games in random matching and constant pairing set-ups respectively. Neither study finds strong differences between stable and unstable games. In a quite different context, Anderson et al. (2004) find that prices diverge from competitive equilibrium that is predicted to be unstable by the theory of tatonnement. Cyclical behavior follows.

## 2. RPSD Games and Theoretical Predictions

The games that were used in the experiments are, firstly, a game we call the unstable RPSD game

$RPSD_U =$

| | R | P | S | D |
|---|---|---|---|---|
| Rock | 90, 90 | 0, 120 | 120, 0 | 20, 90 |
| Paper | 120, 0 | 90, 90 | 0, 120 | 20, 90 |
| Scissors | 0, 120 | 120, 0 | 90, 90 | 20, 90 |
| Dumb | 90, 20 | 90, 20 | 90, 20 | 0, 0 |

and secondly, the stable RPSD game,

$RPSD_S =$

| | R | P | S | D |
|---|---|---|---|---|
| Rock | 60, 60 | 0, 150 | 150, 0 | 20, 90 |
| Paper | 150, 0 | 60, 60 | 0, 150 | 20, 90 |
| Scissors | 0, 150 | 150, 0 | 60, 60 | 20, 90 |
| Dumb | 90, 20 | 90, 20 | 90, 20 | 0, 0 |

Both games are constructed from the well-known Rock-Paper-Scissors game with the addition of a fourth strategy Dumb (called so, as it is never a pure best response). Games of this type were first introduced by Dekel and Scotchmer (1992). Both these games have the same unique Nash equilibrium which is symmetric and mixed with frequencies $p^* = (1,1,1,3)/6$. Thus, while the fourth strategy is "Dumb," it is by far the most frequent in equilibrium. Expected equilibrium payoffs are 45 in both games.

While these two games are apparently similar, they differ radically in terms of predicted learning behavior. To summarize our basic argument, suppose there is a population of players who are repeatedly randomly matched to play one of the two games. Then, if all use a fictitious play like learning process to update their play, in the second game there would be convergence to the Nash equilibrium.[3] In the first game, however, there will be divergence from equilibrium and play will approach a cycle in which no weight is placed on the strategy Dumb (D).

### 2.1 Learning Under Fictitious Play

We state and prove results on the stability of the mixed equilibria in $RPSD_U$ and $RPSD_S$ in Appendix A. Here we give a heuristic account. For example, the outcome in $RPSD_U$ is illustrated in Figure 1, with the red triangle on the base of the pyramid being the attracting cycle. (This is simplex of possible mixed strategies over the four available actions.) This cycle was named a Shapley triangle or polygon after the work of Shapley (1964) who was the first to produce an example of non-convergence of learning in games. See also Gaunersdorfer and Hofbauer (1995) for a detailed treatment.

More recently, Benaïm, Hofbauer and Hopkins (BHH) (2009) observe the following. Suppose people learn according to "weighted" instead of classical fictitious play. Under what we call classical beliefs, a simple average is taken over all observations. Under weighted fictitious play, players construct their beliefs about the play of others by placing greater weight on more recent experience. Then, play in the unstable game will still converge to the Shapley triangle, but, furthermore, the time average of play will converge to a point that they name the TASP (Time Average of the Shapley Polygon), denoted "T" on Figure 1. This is in contrast to Shapley's original result, where in the unstable case nothing converges. For the game $RPSD_U$, the TASP places no weight on the strategy D, despite its weight of 1/2 in Nash equilibrium. That is, it is clearly distinct from the Nash equilibrium of the game, denoted "N" in Figure 1.

However, it is not the case that theory predicts that the frequency of *D* should decrease

[3]Fictitious play is perhaps the most enduring model of learning in games. See Fudenberg and Levine (1998, Chapter 2) for an introduction. There are, of course, other learning models. Young (2004) gives a survey of recent developments.

monotonically. Specifically, (see Proposition 2 in Appendix A) we identify a region $E$ in the space of mixed strategies where $D$ is the best response and so its frequency will grow. This region $E$ is a pyramid within the pyramid in Figure 1, with the Shapley triangle as its base and apex at the Nash equilibrium. But under fictitious play, given almost all initial conditions, play will exit $E$ and the frequency of $D$ will diminish.

In the second game $RPSD_S$, by contrast, the mixed equilibrium is stable under most forms of learning, including fictitious play. Hence, one would expect to see the average frequency of the fourth strategy, $D$, to be close to one half.

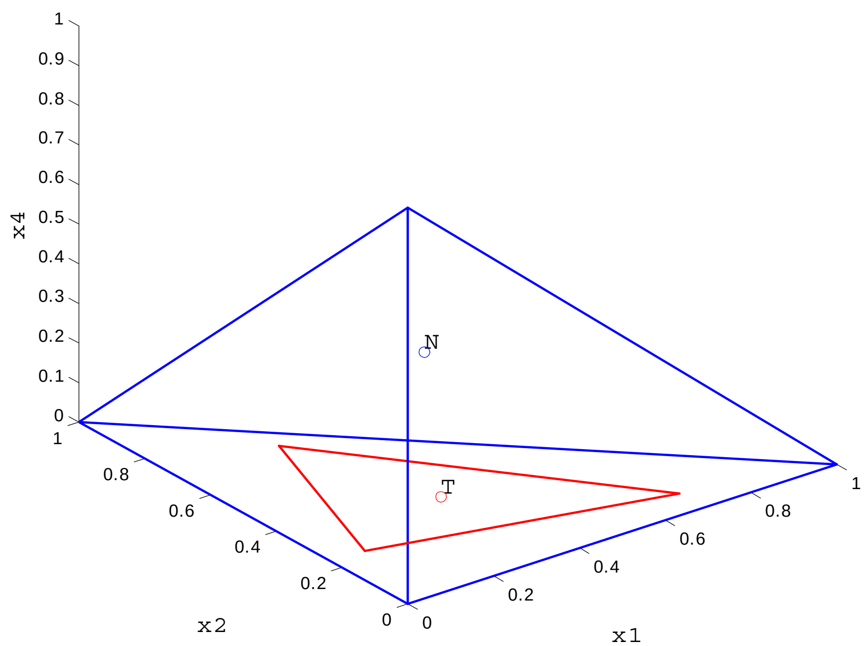



**Figure 1: Nash equilibrium (N) and TASP (T) in the unstable version of the *RPSD* game. The frequencies of strategies 1 and 2 are on the horizontal axes and of strategy 4 on the vertical axis.**

Thus, if fictitious play describes agents' behavior, the limiting frequency of $D$ is a ready indicator of whether we are in the stable or unstable case. It is also, therefore, a simple way to determine whether the predictions of fictitious play, and learning theory, hold in practice.

Equilibrium theory suggests that the frequency of $D$ should be the same in both games. Learning theory suggests they should differ.

### 2.2 Noisy Play: SFP and QRE

This clean distinction is unlikely to occur exactly in practice, given that actual behavior is often noisy. This tendency for subjects to make mistakes or to experiment can be captured theoretically in two linked ways. The first is to move to stochastic fictitious play (SFP), a modification of the original learning model that allows for random choice. The second is to look at perturbed equilibria known as quantal response equilibria (QRE). The connection is that QRE are the fixed or stationary points for the SFP learning process.

The stochastic choice rule that is most frequently used is the logit version where the probability of player $i$ taking action $j$ of a possible $n$ at time $t$ is given by

$$p_i^j(t) = \psi^j(A_i(t)) = \frac{e^{\lambda A_i^j(t)}}{\sum_{k=1}^{n} e^{\lambda A_i^k(t)}}$$

where $\lambda \geq 0$ is a precision parameter and in SFP the attraction $A_i^j(t)$ will be the expected payoff to action $j$ at time $t$. As $\lambda$ becomes large, the probability of choosing the action with the highest expected payoff, the best response, goes to one.

In fictitious play, expectations about payoffs are derived from expectations over opponents' actions which in turn are derived from past observations of play. Recently, in the experimental literature, the formation of beliefs has most often been approached in the context of the EWA (experience weighted attraction) model of Camerer and Ho (1999). This is a more general specification which includes SFP, both the weighted form and with classical beliefs, and other learning models as special cases. Attractions in the EWA model are set by

$$A_i^j(t) = \frac{\phi N(t-1)A_i^j(t-1) + [\delta + (1-\delta)I_j(t-1)]\pi_j(t-1)}{N(t)}$$

where $N(t) = \rho N(t-1) + 1$, and $\phi$ and $\rho$ are recency parameters. For classical beliefs, $\rho = \phi = 1$; for weighted beliefs $\rho = \phi < 1$. The parameter $\delta$ is an imagination factor and for all forms of fictitious play, it is set to 1, and $I_j(t-1)$ is an indicator function that is one if action $j$

is chosen at $t-1$ and is zero otherwise.[4] Finally, $\pi^j$ is the (implied) payoff to strategy $j$. In this context, we deal with simple strategic form games, so that given a game matrix $B$, we will have the payoff to strategy $j$ being $\pi^j = B_{jk}$ given the opponent chooses action *k*.

Equilibrium in SFP occurs when the expected payoffs are consistent with players' actual choice frequencies. This idea for a perturbed equilibrium was proposed independently by McKelvey and Palfrey (1995) under the title of QRE. Given the logit choice rule, one finds the QRE equilibrium frequencies by solving the following system of equations

$$p^j = \psi^j(\pi(p)) = \frac{e^{\lambda\pi^j(p)}}{\sum_{k=1}^n e^{\lambda\pi^k(p)}} \text{ for } j = 1,...,n$$

where $\pi(p) = B.p$ with *B* being the payoff matrix. QRE with the specific logit choice rule can also be called logit equilibrium (LE).

The principal method for the analysis of SFP has been by way of the perturbed best response dynamics

$$\dot{x} = \psi(\pi(x)) - x \qquad \textbf{(PBR)}$$

where the function $\psi(\cdot)$ is a perturbed choice function such as the logit above and *x* is a vector of players' beliefs. Results from stochastic approximation theory show that a perturbed equilibrium is locally stable under SFP if it is stable under the perturbed dynamics. See Benaïm and Hirsch (1999), Hopkins (1999) and Ellison and Fudenberg (2000) for details.

One well-known property of QRE is that as the precision parameter $\lambda$ increases in value, the set of QRE approaches the set of Nash equilibria. But notice that given the logit formulation above an increase in λ is entirely equivalent to an increase in payoffs. For example, if all payoffs are doubled this would have the same effect as doubling $\lambda$.

Specific results for logit equilibria in the games $RPSD_U$ and $RPSD_S$ are the following:

1) Each logit equilibrium is of the form $\hat{p} = (m, m, m, k)$ where $k = 1 - 3m$ and is unique for a given value of $\lambda$. That is, each LE is symmetric in the first three strategies.
2) The value of $k$, the weight placed on the fourth strategy D, is in $[1/4, 1/2)$ and is strictly increasing in $\lambda$. That is, the LE is always between the Nash equilibrium $(1,1,1,3)/6$ and uniform mixing $(1,1,1,1)/4$ and approaches the Nash equilibrium as $\lambda$ or

[4]The EWA model permits $\rho \neq \phi$ and/or $\delta < 1$ to capture a variety of reinforcement learning approaches.

payoffs become large.

3) For a given value of $\lambda$, the LE of $RPSD_U$ and of $RPSD_S$ are identical. That is, while the level of optimization affects the LE, the stability of the equilibrium does not.

The implications of an increase of the precision parameter $\lambda$, or equivalently of an increase in payoffs, for learning outcomes are quite different. First, it is well known that the stability of mixed equilibria under the perturbed best response (PBR) dynamics depend upon the level of $\lambda$. When $\lambda$ is very low, agents randomize almost uniformly independently of the payoff structure and a perturbed equilibrium close to the center of the simplex will be a global attractor. This means that even in the unstable game $RPSD_U$, the mixed equilibrium will only be unstable under SFP if $\lambda$ is sufficiently large. For the specific game $RPSD_U$, it can be calculated that the critical value of $\lambda$ is approximately 0.17. In contrast, in the stable game $RPSD_S$, the mixed equilibrium will be stable independent of the value of $\lambda$.

This is illustrated in Figure 2. The smooth red curve, labeled "Stable," gives the asymptotic level of the proportion of the fourth strategy $D$ for game $RPSD_S$ under the perturbed best response (PBR) dynamics as a function of $\lambda$. The smooth blue curve, labeled "Unstable," gives the asymptotic level of the proportion of the fourth strategy $D$ for game $RPSD_U$. For low values of $\lambda$, that is on the interval [0, 0.17], the perturbed best response dynamics converge to the LE in both games. Indeed, in the stable case, the dynamics always converge to the LE and this is why the red "Stable" curve thus also gives the proportion of $D$ in the LE as a function of the precision parameter $\lambda$.

However, the behavior in $RPSD_U$ is quite different for values of $\lambda$ above the critical value of 0.17. The LE is now unstable and from almost all initial conditions play converges to a cycle. But given the presence of noise this cycle can be quite different from the Shapley triangle illustrated in Figure 1 (basically, the noise ensures that every strategy is played with positive probability). Only as $\lambda$ increases does the proportion of $D$ approach zero. This leads to very different comparative static outcomes. An increase in $\lambda$ for game $RPSD_S$ leads to an increase in the frequency of $D$. However, for $\lambda$ greater than 0.17, an increase in either $\lambda$ or in payoffs should lead to a decrease in the frequency of $D$. Remember that the prediction of QRE/LE is that an increase in payoffs or $\lambda$ should lead to an increase in the weight on $D$ in both games.

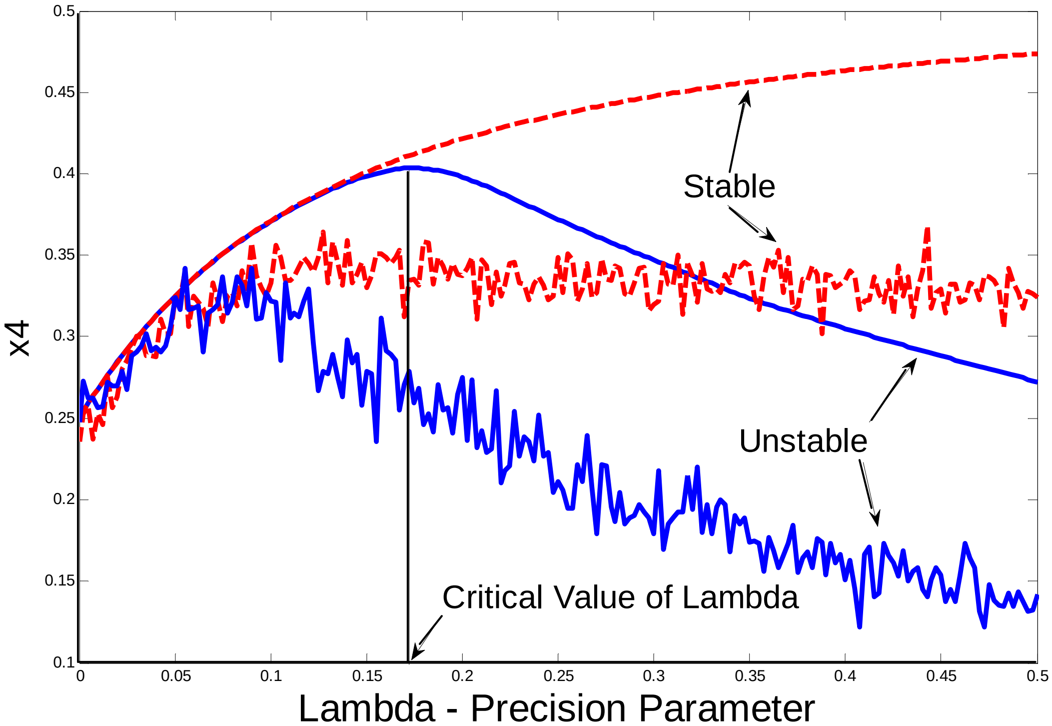



**Figure 2: Frequencies of the 4th strategy D against λ the precision parameter. The smooth lines are generated by continuous time learning processes, the jagged lines by simulations of the experimental environment. Stable/Dashed refers to the $RPSD_S$ game, Unstable/Solid to the $RPSD_U$ game. The smooth Stable line also gives the frequency of the 4th strategy in the logit equilibrium as a function of λ for both games.**

There are several qualifications in applying these theoretical results to an experimental setting. In effect, the theoretical framework assumes an infinite population of agents all who share the same beliefs and investigates asymptotic behavior, taking the further limit of the recency parameter $\rho$ to one. In the experiments we must work with a finite population and a finite horizon. We, therefore, also report simulations of populations of twelve agents who play 80 repetitions (both values chosen to be the same as the experiments we run). Each simulated agent learns according to weighted SFP (that is, EWA with $\phi = \rho < 1$ and $\delta = 1$). We set the recency parameter $\phi = \rho = 0.8$ and then vary the precision parameter $\lambda$. We ran one simulation for each value of $\lambda$ in the sequence 0, 0.0025, 0.005, ..., 0.5 for each of the two games $RPSD_U$ and $RPSD_S$. Initial conditions in each simulation were set by taking the initial attractions to be drawn from a uniform distribution on [0, 150]. The resulting average levels of the frequency of $D$ over the whole 80 periods and 12 simulated subjects are graphed as jagged lines in Figure 2. As learning outcomes over a finite horizon are stochastic, there is considerable variation from one simulation to the next even though the value of $\lambda$ changes slowly. What is encouraging, however, is that the simulations preserve the same qualitative outcomes as the asymptotic results generated by the theoretical models.[5] The experiment employed four treatments in a $2 \times 2$ design. First, the game was varied between $RPSD_U$ and $RPSD_S$. Second, payoffs were in experimental francs and we varied the rate of exchange. In high payoff treatments the rate of exchange was 2.5 times higher than in low payoff treatments. Theoretically, as noted above, this is the same as an increase in $\lambda$. An empirical effect of this type is reported in Battalio et al. (2001), though the increase is less than one-for-one. Given the theoretical arguments outlined above, reinforced by the simulations, we would expect the following outcomes.

### 2.3 Testable Hypotheses

1) **Nash Equilibrium (NE):** average play should be at the NE $(1,1,1,3)/6$ in all treatments.
2) **Quantal Response/Logit Equilibrium (LE):** play should be between NE and $(1,1,1,1)/4$ in both stable and unstable treatments, but play should be closer to Nash equilibrium, and the proportion of $D$ higher, in high payoff treatments.
3) **TASP:**
   a) play should be closer to the TASP in unstable treatments, but closer to LE in stable treatments
   b) play should be closer to the TASP in the high payoff unstable treatment (smaller proportion $D$) than in the low payoff unstable treatment, but play should be closer to Nash equilibrium in the high payoff stable treatment (higher proportion $D$) than in the low payoff stable treatment.
   c) Average play should converge in all treatments but in the unstable treatments beliefs

[5] Clearly, however, they are not identical. Further simulations, not reported here, indicate that this difference cannot be ascribed to any one of the three factors (finite horizon, finite population, ρ<1) but rather arises from a combination of the three.

should continue to cycle.

## 3. Experimental Design and Procedures

The experiment featured a full factorial two-by-two design. One treatment variable was the game payoff matrix, either the unstable game $RPSD_U$ or the stable game $RPSD_S$ shown earlier. The other treatment variable was the payoff conversion rate of Experimental Francs (EF, the entries in the game matrix) to U.S. Dollars. In the High Payoffs treatment, 100 EF = \$5. In the Low Payoffs treatment, 100 EF = \$2. Subjects also received an extra, fixed "participation" payment of \$10 in the Low Payoffs treatment to ensure that their total earnings comfortably exceeded their opportunity cost.

Each period each player $i$ entered her choice $s_i^j$ = 1, 2, 3, or 4 (for Rock, Paper, Scissors, Dumb), and at the same time entered her beliefs about the opponent's choice in the form of a probability vector ($p_1$, $p_2$, $p_3$, $p_4$). When all players were done, the computer matched the players randomly into pairs and announced the payoffs in two parts. The game payoff was obtained from the matrix, and so ranged from 0 to 120 or 150 EF. The prediction payoff was $5-5\sum_{i=1}^{4} p_i^2 + 10 p_j$ when the opponent's actual choice was $j$, and so ranged from 0 to 10 EF.

The payoff scheme was chosen because belief data allow diagnostic tests of the competing models, and because belief elicitation itself can help focus players on belief learning (Ruström and Wilcox, 2006). The quadratic scoring rule was calibrated so that the prediction payments were an order of magnitude smaller than the game payoffs, reducing the incentive to hedge action choices by biasing reported beliefs.[6]

In each session, 12 subjects were randomly and anonymously re-matched over a computer network for a known number of 80 periods to play the same game, either $RPSD_U$ or $RPSD_S$.[7] After each period, subjects learned the action chosen by their opponent, their own

[6] This potential for biased beliefs does not appear to be empirically significant in practice, at least as measured for other games (Offerman et al., 1996; Sonnemans and Offerman, 2001). Taken by itself, the quadratic scoring rule is incentive compatible (Savage, 1971), and is commonly used in experiments with matrix games (e.g., Nyarko and Schotter, 2002).

[7] Some experiments studying learning and stability in games have used a longer 100 or 150 period horizon (e.g., Tang, 2001; Engle-Warnick and Hopkins, 2006). We used the shorter 80-period length because subjects needed to input beliefs and this lengthened the time to complete each period. Including instructions and payment time, each session lasted about two hours. One of the 12 sessions was unexpectedly shortened to 70 periods due to a move by nature: a tornado warning that required an evacuation of the experimental laboratory.

payoffs, as well as the frequency distribution of actions chosen by all 12 subjects in the session. At the conclusion of the session, 10 of the 80 periods were drawn randomly without replacement for actual cash payment using dice rolls (to control for wealth effects).

We conducted three sessions in each of the four treatment conditions, for a total of 144 subjects. Two sessions in each treatment were conducted at Purdue University, and one session in each treatment was conducted at UC-Santa Cruz. All subject interaction was computerized using z-Tree (Fischbacher, 2007). The experiment employed neutral terminology, such as "the person you are paired with" rather than "opponent" or "partner." Action choices were labeled as A, B, C and D, and the instructions and decision screens never mentioned the words "game" or "play." The instructions in Appendix B provide additional details of the framing, and also show the decision and reporting screens.

## 4. Experiment Results

We begin with a brief summary of the overall results before turning to more detailed analysis. Figures 3 and 4 show the cumulative proportion of action choices for two individual sessions out of the 12 conducted for this study. Figure 3 displays a session with the unstable matrix and high payoffs. Paper and Scissors are initially the most common actions. Scissors appears to rise following the early frequent play of Paper, followed by a rise in the frequency of Rock. This pattern is consistent with simple best response dynamics. Dumb is played less than a quarter of the time until the second half of the session and its rate tends to rise over time. Figure 4 displays a session with the stable matrix and high payoffs. The Paper, Scissors and Rock rates again fluctuate, in the direction expected by best response behavior. For the stable matrix, Dumb starts at a higher rate and rises closer to the Nash equilibrium prediction of 0.5 by the end of the session.

Figure 5 and Table 1 provide a pooled summary for all 12 sessions. The figure displays the cumulative frequency that subjects play the distinguishing Dumb action in each of the four treatments. This rate tends to rise over time, but is always lowest in the unstable, high payoffs condition as predicted by the TASP model. Table 1 shows that Dumb is played about 26 percent of the time overall in this treatment, compared to about 40 percent in the stable matrix treatments.

**Figure 3: Example Unstable Session-Cumulative Proportion Choosing Each Action (Session 5: High Payoffs, Unstable Matrix)**

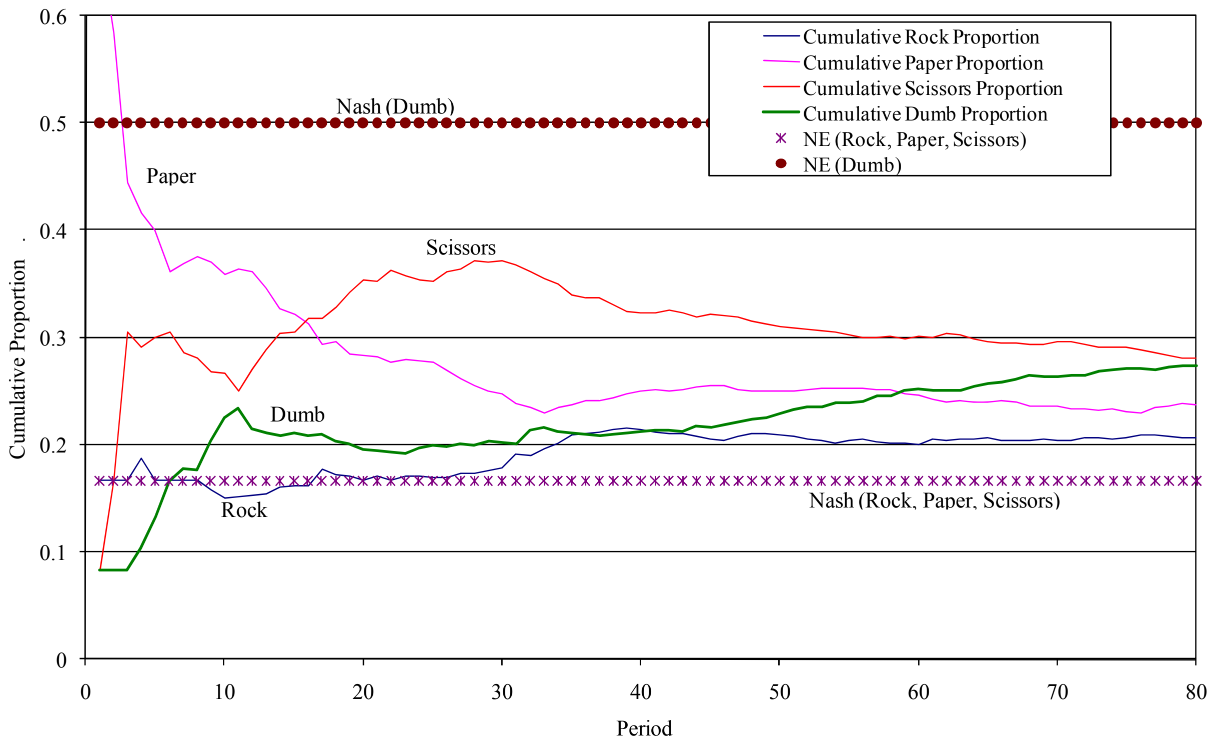



**Figure 4: Example Unstable Session-Cumulative Proportion Choosing Each Action (Session 11: High Payoffs, Stable Matrix)**

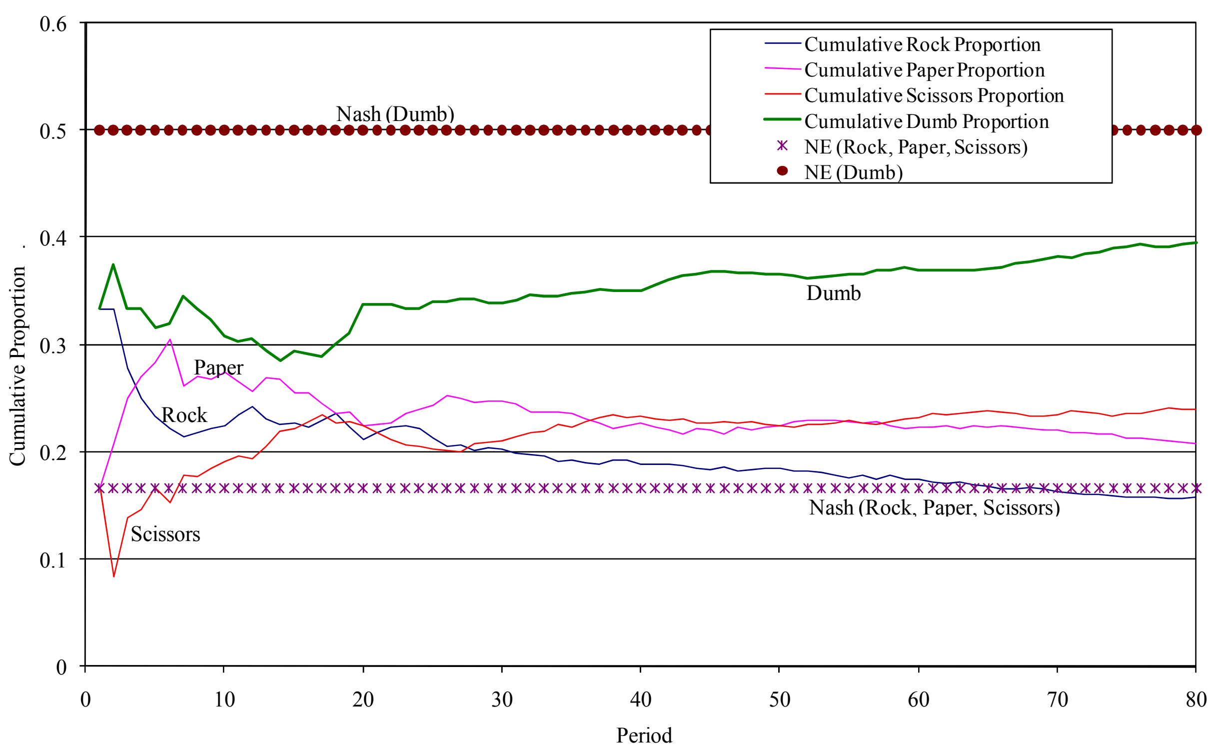


**Figure 5: Cumulative Proportion Choosing Action Dumb, Over Time for All Treatments , All 12 Sessions**

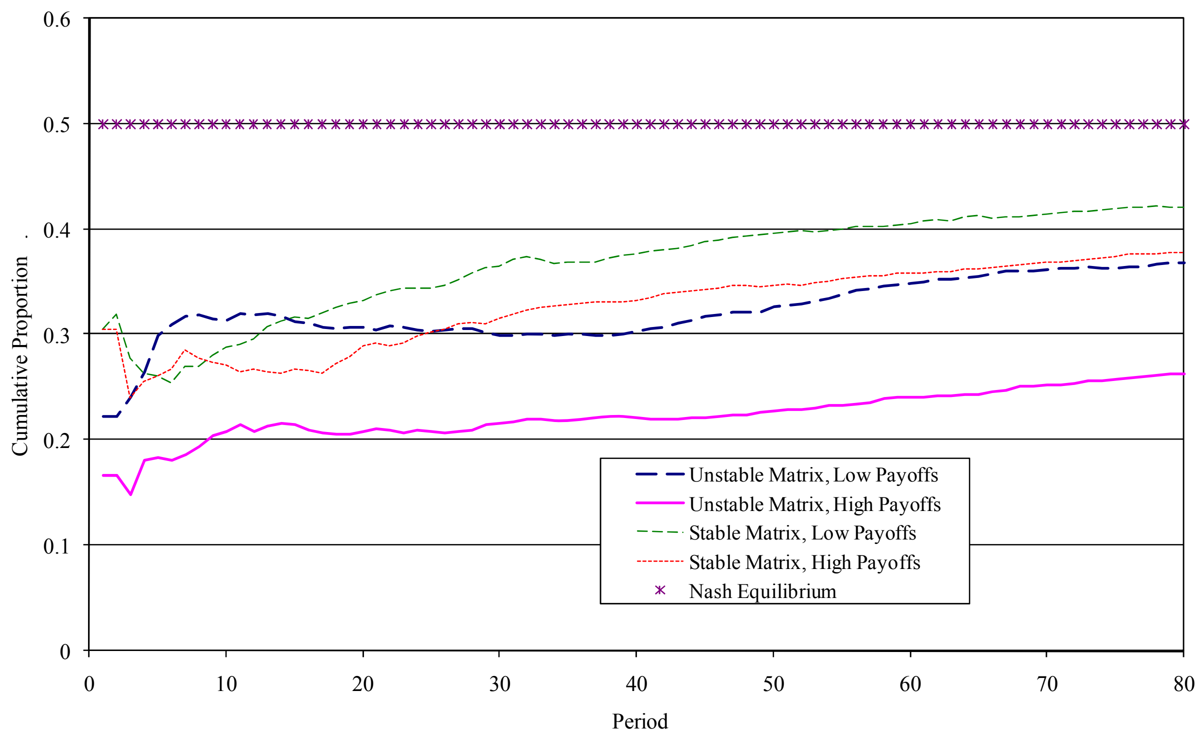



**Table 1: Theoretical Predictions and Observed Frequencies of Each Action for Each Treatment Condition**

| | Frequencies | | | |
|---|---|---|---|---|
| **Theory** | **Rock** | **Paper** | **Scissors** | **Dumb** |
| - Nash | 0.167 | 0.167 | 0.167 | 0.5 |
| - QRE | [0.25, 0.167] | [0.25, 0.167] | [0.25, 0.167] | [0.25, 0.5] |
| - TASP | 0.333 | 0.333 | 0.333 | 0 |
| **Results** | | | | |
| Unstable, High payoffs | 0.226 | 0.231 | 0.280 | 0.263 |
| Unstable, Low payoffs | 0.221 | 0.203 | 0.207 | 0.368 |
| Stable, High payoffs | 0.176 | 0.233 | 0.212 | 0.378 |
| Stable, Low payoffs | 0.172 | 0.204 | 0.204 | 0.420 |

### 4.1 Tests of the Principal Hypotheses

Figure 5 indicates an upward time trend in all treatments for the rate that subjects choose the critical action Dumb. The Dumb action is played more frequently over time even in the unstable game $RPSD_U$. Although the research questions of this study principally concern learning and game play over a very long horizon, for practical reasons the experiment sessions included only 80 periods of play. Nevertheless, we can draw some statistical inferences about long-run, asymptotic play using this 80-period time series.

We focus on the following reduced form model of subjects' choice of the critical strategy Dumb,

$$y_{it}^* = \sum_{j=1}^{3} \beta_{1j} D_j (1/t) + \beta_2((t-1)/t) + u_i + v_{it},$$

$$y_{it} = 1 \text{ if } y_{it}^* > 0 \text{ and 0 otherwise.}$$

The $t$, $i$ and $j$ subscripts index the period, subject and session, and the $D_j$ are dummy variables that have the value of 1 for the indicated session within each treatment. We assume logistically-distributed errors $u_i + v_{it}$, including a random effect error component for subject $i$ ($u_i$), so this can be estimated using a random effects logit model. [This panel data approach accounts for the repeated choices made by the same individual subjects and the resulting non-independence of actions within subjects and within sessions.] Note that the time index $t$=1 in the first period, so that ($t$-1)/$t$ is zero. Thus the $\beta_{1j}$ coefficient provides an estimate of the probability for choosing Dumb at the start of session $j$. As $t\rightarrow\infty$ the 1/$t$ terms approach 0 while the ($t$-1)/$t$ term approaches one, so the $\beta_2$ coefficient provides an estimate of the asymptotic probability of choosing Dumb in the treatment. All three models discussed in Section 2 (Nash, QRE, TASP) predict stable long-run rates of Dumb play, although play of the other actions continues to cycle in TASP. A similar reduced form empirical specification has been employed to model convergence behavior in dozens of experimental studies since Noussair et al. (1995).[8]

The Dumb strategy is played half the time in the Nash equilibrium, which implies the null hypothesis of $\beta_2$=0 since the logit model probability $F(x)$=exp($x$)/[1+exp($x$)] is 0.5 at $x$=0. Table 2 presents the estimation results for the asymptote $\beta_2$ coefficients. Only the low payoffs,

[8] Qualitatively identical conclusions result from an alternative specification that does not use the 1/$t$ and (($t$-1)/$t$) terms to model early and late period play, but instead simply uses treatment dummy variables and time trends in random effect logit models to compare the likelihood of the Dumb action across treatments. We also draw the same conclusions under the assumption of normally-distributed errors, estimating this equation using a probit model.

stable game asymptotic estimate is not significantly different from 0. This indicates that the Dumb strategy is converging toward the Nash equilibrium rate of 0.5 only for the low payoffs, stable game treatment. The data thus reject the Nash equilibrium Hypothesis 1 for three out of the four treatments. The data also reject the Quantal Response/Logit Equilibrium Hypothesis 2, since according to this hypothesis play should be closer to Nash in the high payoff treatments. This implies coefficient estimates closer to 0 for high payoffs compared to low payoffs, but columns (1) and (2) display the opposite pattern. Moreover, column (5) shows that the differences are statistically significant, in a direction contrary to Hypothesis 2, for the pooled data and for the unstable game. That is, play is further away from Nash in the high payoff treatments.

The coefficient estimates are consistent only with the TASP Hypothesis 3. The estimated asymptotic rate that subjects pay Dumb is further from the Nash equilibrium and closer to the TASP prediction for the unstable treatments and for high payoffs. The $\hat{\beta}_2$ = -1.282 estimate for the unstable, high payoff treatment implies a point estimate of a 22 percent rate for the Dumb strategy. While this rate is below the Nash equilibrium, it is also well above the rate of 0 predicted by TASP. Thus, data are consistent with only the comparative statics predictions of TASP, and not its quantitative implications.

Average payoffs in the Nash equilibrium are 45, while average payoffs are 90 over the entire Shapley cycle. Thus average payoffs provide another measure to distinguish TASP and Nash. The average payoffs per player mirror the pattern of the Dumb strategy frequency, both in the trend towards Nash over time and in the ranking across treatments. The following cross-sectional regression of each subject's average earnings per period indicates that the marginal impact of switching from the unstable to the stable game is about a 10 percent decrease in payoffs. It also shows that Experimental Franc payoffs are about 10 percent higher in the high conversion rate treatment compared to the low conversion rate treatment. Both effects are significant at the 5 percent level.[9]

| | | | | |
|---|---|---|---|---|
| Ave (EF) Payoff = | 54.5 | – 5.41(RPSD$_S$ dummy) | + 4.94(High Payoff dummy) | $N$=144 |
| (std. errors) | (1.8) | (2.20) | (2.20) | $R^2$=0.17 |

[9] This regression accounts for across-subject correlation of payoffs within sessions by adjusting standard errors to account for session clustering. The average payoffs do not include earnings from belief accuracy.

**Table 2: Random Effects Logit Model of Dumb Action Choice**

**Dependent Variable = 1 if Dumb Chosen; 0 otherwise**

| **Estimation Dataset** | **High Payoff× (*t*-1)/*t* (1)** | **Low Payoff× (*t*-1)/*t* (2)** | **Unstable× (*t*-1)/*t* (3)** | **Stable× (*t*-1)/*t* (4)** | **Probability (coefficients equal)[a] (5)** | **Obser-vations** | **Subjects** | **Log-L** |
|---|---|---|---|---|---|---|---|---|
| All Sessions | -0.909** (0.210) | -0.408* (0.209) | | | 0.046 | 11400 | 144 | -5470.6 |
| All Sessions | | | -0.946** (0.209) | -0.370[†] (0.208) | 0.026 | 11400 | 144 | -5470.1 |
| Unstable Only | -1.282** (0..291) | -0.611* (0.289) | | | 0.050 | 5640 | 72 | -2596.5 |
| Stable Only | -0.537[†] (0.295) | -0.205 (0.294) | | | 0.213 | 5760 | 72 | -2872.0 |
| High Payoffs Only | | | -1.283** (0.292) | -0.537[†] (0.290) | 0.035 | 5640 | 72 | -2543.5 |
| Low Payoffs Only | | | -0.612* (0.294) | -0.205 (0.293) | 0.165 | 5760 | 72 | -2925.0 |

Note: Session×(1/t) dummies included in each regression are not shown. Standard errors in parentheses. [†] indicates coefficient significantly different from 0 at the 10-percent level; * indicates coefficient significantly different from 0 at the 5-percent level; ** indicates coefficient significantly different from 0 at the 1-percent level (two-tailed tests). [a]One-tailed likelihood ratio tests, as implied by the TASP research hypotheses.

### 4.2 Learning and Stochastic Best Responses

Average play moves closer to the Nash equilibrium frequencies over time. This is not anticipated by learning theory for the unstable game, particularly for the high payoff treatment. In order to help understand the possible learning process employed by the subjects, we empirically estimate the EWA learning model that was presented in Section 2.

In our application, the realized (or forgone) profit $\pi_i(t\text{-}1)$ is calculated based on the observed action chosen by the paired player in the previous period. Our implementation of the model incorporates stochastic best responses through a logit choice rule, the same as typically used in QRE applications. For our application with 80 periods, 36 subjects per treatment and 4 possible actions, the log-likelihood function is given by

$$LL(A(0), N(0), \phi, \rho, \delta, \lambda) = \sum_{t=1}^{80} \sum_{i=1}^{36} \ln\left( \sum_{j=1}^{4} I(s_i^j, s_{-i}(t)) \cdot P_i^j(t) \right),$$

where $I$ is an indicator function for the subjects' choice and $P_i^j(t)$ is player $i$'s probability of choosing action $j$.

Table 3 reports the maximum likelihood estimates for this model.[10] The estimated decay parameter $\phi$ always exceeds the decay parameter $\rho$, and in all four treatments a likelihood ratio test rejects the null hypothesis that they are equal. Nevertheless, we also report estimates for a restricted model on the right side of that table which imposes restrictions of $\phi = \rho$ and $\delta = 1$ to implement the special case of weighted stochastic fictitious play.[11] The large increase in the estimated log-likelihood, however, indicates that the data strongly reject those restrictions.

A drawback of the estimation results shown on the left side of Table 3 is that they pool across subjects whose learning could be heterogeneous. Wilcox (2006) shows that this heterogeneity can potentially introduce significant bias in parameter estimates for highly nonlinear learning models such as EWA. He recommends random parameter estimators to address this problem, and with his generous assistance we are able to report such estimates in the center of Table 3. The assumed distributions are lognormal for $\lambda$ and a transformed normal

[10] We impose the initial conditions $A(0)$=1 and $N(0)$=0 for all four strategies, but the results are robust to alternative initial attraction and experience weights.

[11] The restriction of $\delta = 1$ implies that a subject's unchosen actions receive the same weight as chosen actions in her belief updating, which is the assumption made in fictitious play learning.

**Table 3: Experience-Weighted Attraction and Stochastic Fictitious Play Learning Model Estimates**

| | EWA | | | | | EWA *random parameter estimates* | | | | | Weighted Stochastic Fictitious Play $\phi$=$\rho$, $\delta$=*1* | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Unstable, Low Payoffs | Unstable, High Payoffs | Stable, Low Payoffs | Stable, High Payoffs | | Unstable, Low Payoffs | Unstable, High Payoffs | Stable, Low Payoffs | Stable, High Payoffs | | Unstable, Low Payoffs | Unstable, High Payoffs | Stable, Low Payoffs | Stable, High Payoffs |
| Decay Parameters | | | | | | | | | | | | | | |
| $\phi$ | 0.889 | 0.882 | 0.910 | 0.934 | E($\phi$) | 0.878 | 0.873 | 0.834 | 0.879 | $\phi$ | 0.997 | 0.828 | 1.000 | 1.000 |
| | (0.029) | (0.024) | (0.012) | (0.009) | CV($\phi$) | 0.193 | 0.171 | 0.152 | 0.175 | | (0.024) | (0.111) | | |
| | | | | | median $\phi$ | 0.950 | 0.932 | 0.869 | 0.940 | | | | | |
| $\rho$ | 0.568 | 0.488 | 0.634 | 0.529 | E($\rho$) | 0.871 | 0.870 | 0.702 | 0.870 | $\rho$ | | | | |
| | (0.171) | (0.155) | (0.075) | (0.205) | CV($\rho$) | 0.192 | 0.171 | 0.152 | 0.175 | | | | | |
| | | | | | median $\rho$ | 0.942 | 0.927 | 0.731 | 0.930 | | | | | |
| Imagination Factor | | | | | | | | | | | | | | |
| $\delta$ | 0.000 | 0.000 | 0.000 | 0.000 | $\delta$ | 0.072 | 0.114 | 0.000 | 0.003 | $\delta$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | (Likelihood maximized at 0 bound) | | | | | (0.060) | (0.075) | (0.000) | (0.052) | | (Constrained at 1) | | | |
| Payoff sensitivity | | | | | | | | | | | | | | |
| $\lambda$ | 0.014 | 0.011 | 0.015 | 0.010 | E($\lambda$) | 0.067 | 0.055 | 0.024 | 0.067 | $\lambda$ | 0.030 | 0.012 | 0.016 | 0.019 |
| | (0.004) | (0.003) | (0.003) | (0.004) | CV($\lambda$) | 0.442 | 0.366 | 0.763 | 0.352 | | (0.006) | (0.005) | (0.003) | (0.004) |
| | | | | | median $\lambda$ | 0.061 | 0.052 | 0.019 | 0.063 | | | | | |
| Log-Like | -3053.5 | -3091.3 | -3055.6 | -3007.6 | | -2978.0 | -3074.5 | -3009.2 | -2985.8 | | -3843.4 | -3732.9 | -3889.6 | -3863.3 |

Notes: Standard errors in parentheses. CV denotes the estimated coefficient of variation for the parameter distribution: standard deviation/mean.

(to range between 0 and 1) for $\phi$ and $\rho$. The table reports the mean, the coefficient of variation (standard deviation/mean) and the median to summarize the estimated distributions of these parameters. The point estimates for $\phi$ are similar to the central tendency of the estimated distributions, but for $\rho$ and $\lambda$ the point estimates are somewhat lower than the estimated distribution means. Although this is consistent with a statistical bias arising from imposing homogeneity, these random parameter estimates do not qualitatively change the puzzling finding that the imagination factor $\delta$ is near zero. That is, subjects' learning evolves as if they focus only on realized payoffs and actions, and not on unchosen actions, contrary to fictitious play learning.

Note also that the payoff sensitivity/precision parameter estimates ($\lambda$) are always quite low, and they never approach the critical level (0.17) identified in the continuous time and simulated learning models (Section 2).[12] The estimates also do not increase systematically with the treatment change from low to high payoffs. This suggests that subjects were not very sensitive to payoff levels and were not more sensitive to payoffs that were 2.5 times higher in the high payoff treatment. In other words, although as predicted subjects played Dumb less frequently in the Unstable/High payoff treatment, the structural estimates of this learning model suggest that they did not respond as expected to this treatment manipulation.

### 4.3 Beliefs and Best Responses

Recall that average play is expected to converge in all treatments, but if the TASP is a reasonable approximation of final outcomes then in the unstable game treatment beliefs should continue to cycle. The difficulty in identifying a cycle is that its period depends on how quickly players discount previous beliefs and their level of payoff sensitivity. As documented in the previous subsection, these behavioral parameters are estimated rather imprecisely and the weighted stochastic fictitious play model is a poor approximation of subject learning for these games. Nevertheless, we can compare whether in later periods beliefs vary more in the unstable game than the stable game.

Table 4 summarizes this comparison using the mean absolute value of subjects' change in their reported belief from one period to the next, for each of the four actions. Although beliefs

[12] In the random parameter estimates shown in the middle of Table 3, the 99th percentile of the estimated lognormal distribution of $\lambda$ is less than 0.17 in all treatment conditions.

change by a smaller amount in the later periods for all treatment conditions, this increase in belief stability is insignificant in the unstable, high payoff treatment. Beliefs change on average by 2 percent less in periods 41-80 compared to periods 1-40 in the Unstable/High treatment. By comparison, beliefs change on average by 24 percent less in periods 41-80 compared to periods 1-40 in the other three treatments. This provides evidence that belief stability improves over time *except* for the Unstable/High payoff treatment.

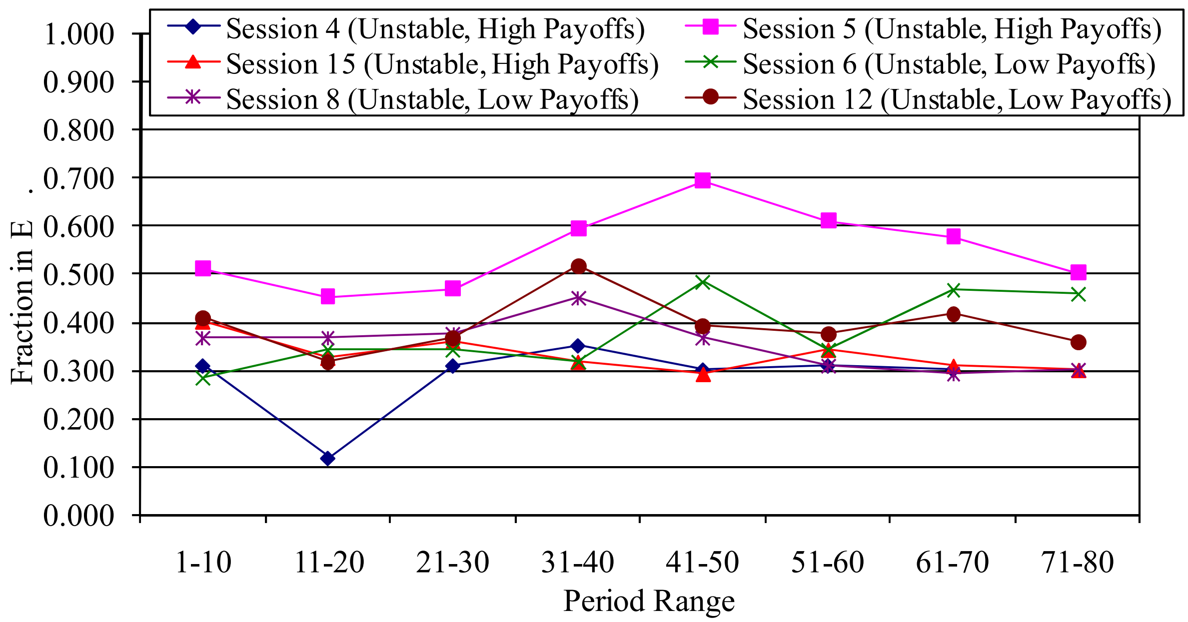
**Table 4: Mean Absolute Change in Reported Beliefs**

| | Unstable, Low Pay | | Unstable, High Pay | | Stable, Low Pay | | Stable, High Pay | |
|---|---|---|---|---|---|---|---|---|
| | Period <41 | Period >40 | Period <41 | Period >40 | Period <41 | Period >40 | Period <41 | Period >40 |
| Rock | 0.147 | 0.107 | 0.117 | 0.128 | 0.111 | 0.087 | 0.133 | 0.091 |
| Paper | 0.148 | 0.093 | 0.129 | 0.130 | 0.135 | 0.097 | 0.132 | 0.094 |
| Scissors | 0.153 | 0.113 | 0.139 | 0.126 | 0.130 | 0.088 | 0.139 | 0.095 |
| Dumb | 0.133 | 0.146 | 0.092 | 0.083 | 0.135 | 0.115 | 0.137 | 0.114 |
| Ave. % reduction In belief change periods 1-40 to periods 41-80 | 20.2% | | 2.2% | | 24.2% | | 27.2% | |

Consider next the relationship between beliefs and best responses. As discussed in Appendix A (Proposition 2), the set of mixed strategies can be partitioned into a set E, for which the best response is Dumb, and everything else (denoted set F). In the unstable game the set E is a pyramid with the Shapley triangle as its base and the Nash equilibrium as its apex. Importantly, the point where all actions are chosen with equal probability is in this pyramid, and for many sessions average play begins (roughly) in this region. Therefore, we might expect the frequency of Dumb to increase initially. But at some point if beliefs move out of E into F then the frequency of Dumb might fall.

Since subjects report their beliefs each period when choosing their action we have a direct measure of when beliefs are in each set. Figure 6 displays the fraction of reported beliefs in set E for each of the six sessions with the unstable game. Although some variation exists across sessions, in most periods between one-third and two-thirds of subjects report beliefs in E. No session shows a substantial downward trend in the fraction of beliefs in E. Subjects do not always best respond to their reported beliefs, particularly when the payoff costs of deviating to another action are small. Nevertheless, we observe subjects in the unstable game choose Dumb 893 out of the 1271 times their reported beliefs are in set E (41.3 percent), and they chose Dumb 893 out of the 2583 times their reported beliefs are in set F (25.7 percent).

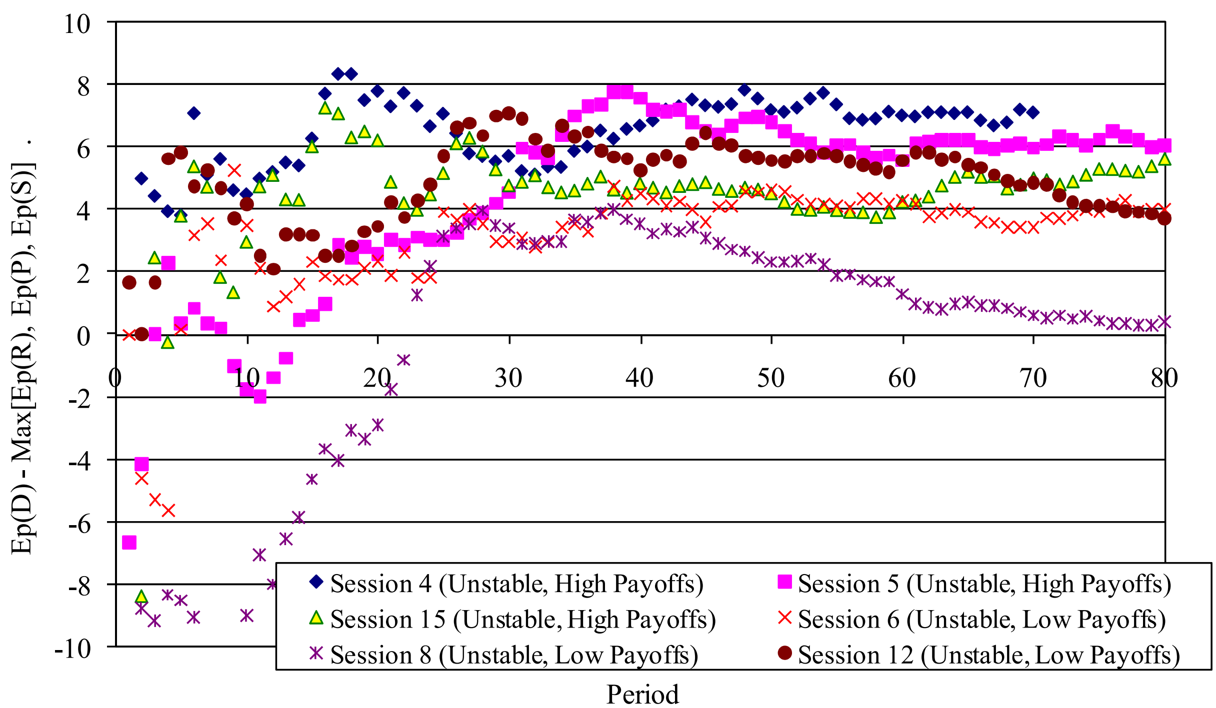
**Figure 6: Fraction of Reported Beliefs in Dumb Best Response Region (Set E)**




The learning model estimates in Table 3 suggest that the belief decay parameter is close to one, particularly when imposing parameter restrictions consistent with weighted stochastic fictitious play ($\phi=\rho$, $\delta=1$). Alternative estimates of the best-fitting decay parameter based directly on reported beliefs (not shown) also indicate a best pooled estimate near one. We also calculated the best-fitting decay parameter for each individual's reported beliefs based on the same procedure employed by Ehrblatt et al. (2007), which minimizes the squared prediction error between the reported belief and the belief implied by the subjects' experience for each possible decay parameter. Constraining this parameter estimate to the interval [0, 1], the best fit is on the boundary of 1 for 92 out of 144 subjects. Thus, a large fraction of our subjects appear to update beliefs in a manner consistent with classical fictitious play.

Based on a maintained assumption of classical fictitious play updating, we can calculate how beliefs would evolve for each session as a function of the cumulative actions of all players. This indicates when aggregate action frequencies suggest beliefs would be in set E or set F. Aggregate beliefs are in set E whenever the expected payoff advantage to action Dumb (compared to the best alternative among Rock, Paper and Scissors) is positive. Figure 7 shows that this payoff advantage is negative (implying aggregate beliefs in set F) in the early periods for about half of the unstable game sessions, but after period 22 the aggregate beliefs based on classical fictitious play are in set E for all 6 sessions. Alternative calculations based on weighted fictitious play and a decay parameter of 0.8, not shown, paint a similar picture overall—except that beliefs move into set F for periods 61-80 of one session (session 8).

**Figure 7: Expected Payoff Advantage to Action Dumb, Based on Cumulative Frequency Distribution of Actions, by Session**



## 5. Discussion

To summarize, the Nash hypothesis fared poorly in our data. The overall rate of playing Dumb ranged from 26 percent in the Unstable/High treatment to 42 percent in the Stable/Low treatment and only began to approach the NE value of one-half towards the end of some Stable sessions. The performance of the QRE (or LE) hypothesis also was unimpressive. Although the observed rates fell into the (rather broad) LE range, the data contradict the main comparative statics prediction: High payoffs led to lower rates of Dumb play, not the predicted higher rates.

The TASP hypothesis had a mixed performance. As predicted, subjects played Dumb least often in the Unstable/High treatment, and most often in the Stable treatments. On the other hand, the proportion of Dumb play showed no consistent tendency to decline over time, much less to zero, in either Unstable treatment.

Some clues can be found in the more detailed examination of the theory and the data. According to theory, learning dynamics in the Unstable treatments should increase the prevalence of Dumb when players' beliefs lie in a tetrahedral subset of the simplex labeled E, and decrease Dumb only when they lie its complement F. The data show that subjects indeed are more likely to play Dumb when they report beliefs in E than in F. However, reported beliefs show little tendency to move (as predicted) into F. Perhaps the reason is that actual play offers little reason for beliefs to move in that direction. In several of the six Unstable sessions, average actual play (the belief proxy in the classic model of learning dynamics, fictitious play) lies in F in the first 20 periods, but it always moves back into E for the remainder of the 80 periods. In one session, with Low payoffs, it gets very close to the boundary of E in the last 20 periods, but in the other sessions Dumb retains a 2 to 8 point advantage in expected payoff.

Another piece of evidence concerns the payoff sensitivity parameter $\lambda$. In theory, there is a critical value, $\lambda \approx 0.17$, below which the TASP prediction fails. That is, for sufficiently low values of $\lambda$, behavior should be similar in Stable treatments as in Unstable treatments: the rate of Dumb play should remain in the 25-40 percent range and be higher in the High payoff treatments.

We estimate the EWA model using aggregate data, and obtain $\lambda$ estimates far below the critical value. This can account for the overall rates of Dumb play. To account for the lower rates of Dumb play in the High payoff treatments, we can point to the tendency of the Unstable simulations in Figure 2 to have a lower proportion of Dumb than the theoretical predictions, even when values of $\lambda$ are relatively low. However, it is also true that the proportion of Dumb play in the Stable treatments is higher, and play is closer to Nash equilibrium, than suggested by the estimated level of $\lambda$.

These accounts, of course, raise further questions. In particular, why do players seem to use such small values of $\lambda$, i.e., respond so weakly to estimated payoff advantages? This weak response to payoffs would appear to be the best explanation for the difference between our experimental results and the point predictions of both equilibrium and learning theory.

One can think of two leading potential explanations for this weak responsiveness. Choosing between them may be the key both to understanding our current results and giving directions for further research. First, payoff differences may have been simply not prominent enough to subjects. In which case, in future experiments, one could improve the feedback or the information provided, perhaps even showing the payoff advantages implied by forecasts and by average play. Second, in contrast, the apparent irresponsiveness of subjects to payoffs in fact indicates that actual subject behavior is only partially captured by the EWA model, even though this model encompasses many forms of learning. Human learning behavior in the complex environment of the current experiments is too sophisticated and too heterogeneous to be captured by current theory, except in terms of the basic comparative statics results that were confirmed here. In this case, the challenge is not to change the experimental design but to provide new and more refined theories of non-equilibrium behavior.

## Appendix A (Stability Properties of RPSD Games)

In this appendix, we state and prove some results on the behavior of the best response (BR) and perturbed best response (PBR) dynamics in the two games $RPSD_U$ and $RPSD_S$. There is already an extensive theoretical literature that shows how the PBR and BR dynamics can be used to predict the behavior of learning under stochastic fictitious play and fictitious play respectively. Specifically, Benaïm and Hirsch (1999), Hopkins (1999b, 2002), Hofbauer and Sandholm (2002) look at the relation between the PBR dynamics and SFP, while Benaïm, Hofbauer and Sorin (2005) show the relationship between the BR dynamics and classical fictitious play. Finally, Benaïm, Hofbauer and Hopkins (2009) look at the relation between the BR dynamics and weighted fictitious play.

We have seen the perturbed best response dynamics (PBR). The continuous time best response (BR) dynamics are given by

$$\dot{x} \in b(\pi(x)) - x \qquad \textbf{(BR)}$$

where $b(\cdot)$ is the best response correspondence.

When one considers stability of mixed equilibria under learning in a single, symmetric population, there is a simple criterion. Some games are positive definite and some are negative definite. Mixed equilibria in positive definite games are unstable, mixed equilibria in negative definite games are stable.

The game $RPSD_U$ is not positive definite. However, the RPS game that constitutes its first three strategies is positive definite. We use this to show that the mixed equilibrium of $RPSD_U$ is a saddlepoint and hence unstable with respect to the BR and PBR dynamics.

**Proposition 1** *In $RPSD_U$, the perturbed equilibrium (LE) $\hat{p}$ is unstable under the logit form of the perturbed best response dynamics for all $\lambda > \lambda^* \approx 0.17$.*

**Proof:** This follows from results of Hopkins (1999b). The linearization of the logit PBR dynamics at $\hat{x}$ will be of the form $\lambda R(\hat{p})B - I$ where $R$ is the replicator operator and $B$ is the payoff matrix of $RPSD_U$. Its eigenvalues will therefore be of the form $\lambda k_i - 1$ where the $k_i$ are the eigenvalues of $R(\hat{p})B$. $R(\hat{p})B$ is a saddlepoint with stable manifold $x_1 = x_2 = x_3$. But for $\lambda$ sufficiently small, all eigenvalues of $\lambda R(\hat{p})B - I$ will be negative. We find the critical value of 0.17 by numerical analysis. □

Further, under the BR dynamics we have converge to a cycle which places no weight on the fourth strategy $D$.

**Proposition 2** *The Nash equilibrium* $p^* = (1,1,1,3)/6$ *of the game* $RPSD_U$ *is unstable under the best response (BR) dynamics. Further, there is an attracting limit cycle, the Shapley triangle, with vertices,* $A_1 = (0.692, 0.077, 0.231, 0)$, $A_2 = (0.231, 0.692, 0.077, 0)$ *and* $A_3 = (0.077, 0.231, 0.692, 0)$, *and time average, the TASP, of* $\tilde{x} = (1,1,1,0)/3$.

**Proof:** We can partition the simplex into two sets. One $E$ is where the best response is the fourth strategy $D$, and $F$ where the best response is one or more of the first three strategies. The set $E$ is a pyramid with base the Shapley triangle on the face $x_4 = 0$ and apex at the mixed strategy equilibrium $p^*$. In $E$, as $D$ is the best response, under the BR dynamics we have $\dot{x}^4 = 1 - x_4 > 0$ and $\dot{x}_i < 0$ for $i = 1,2,3$. If the initial conditions satisfy $x_1 = x_2 = x_3 = (1 - x_4)/3$, then the dynamics converge to $p^*$. Otherwise, the orbit exits $E$ and enters $F$. In $F$, the best response $b$ to $x$ is almost everywhere one of the first three strategies. So we have $\dot{x}_4 < 0$. Further, consider the Liapunov function $V(x) = b \cdot Ax$. We have

$$\dot{V} = b \cdot Ab - b \cdot Ax.$$

As the best response $b$ is one of the first three strategies, we have $b \cdot Ab = 90$ and when $x$ is close to $p^*$, clearly $b \cdot Ax$ is close to the equilibrium payoff of 45. So, we have $V(p^*) = 45$ and $\dot{V} > 0$ for $x$ in $F$ and in the neighborhood of $p^*$. Thus, orbits starting in $F$ close to $p^*$ in fact flow toward the set $b \cdot Ax = 90$, which is contained in the face of the simplex where $x_4 = 0$. The dynamics on this face are the same as for the RPS game involving the first three strategies. One can then apply the results in Benaïm, Hofbauer and Hopkins (2009) to show that the Shapley triangle attracts the whole of this face. So, as the dynamic approaches the face, it must approach the Shapley triangle. Then, the time average can be calculated directly. □

The game $RPSD_S$ is negative definite and hence its mixed equilibrium is a global attractor under both the BR and PBR dynamics. This implies it is also an attractor for (stochastic) fictitious play.

**Proposition 3** *The Nash equilibrium* $p^* = (1,1,1,3)/6$ *of the game* $RPSD_S$ *is globally asymptotically stable under the best response dynamics. The corresponding perturbed equilibrium (QRE) is globally asymptotically stable under the perturbed best response dynamics for all* $\lambda \geq 0$.

**Proof:** It is possible to verify that in the game $RPSD_S$ is negative definite with respect to the set $R_0^n = \{x \in R^n \ : \ \sum x_i = 0\}$. The result then follows from Hofbauer (1995) and Hofbauer and Sandholm (2002). □

What do these results imply for stochastic fictitious play? Suppose we have a large population of players who are repeatedly randomly matched to play either $RPSD_U$ or $RPSD_S$. All players use the choice rule (logit) and updating rule (attract). Assume further that at all times all players have the same information and, therefore, the same attractions. Remember that for SFP we assume that $\rho = \phi$ and that $\delta = 1$.

**Proposition 4** *(a)* $RPSD_U$ *: for* $\lambda > \lambda^* \approx 0.17$, *the population SFP process diverges from the perturbed Nash equilibrium (the logit equilibrium). If* $\rho = \phi < 1$ *, taking the joint limit* $\rho \to 1$, $\lambda \to \infty$ *and* $t \to \infty$, *the time average of play approaches the TASP* $\tilde{p} = (1,1,1,0)/3$.

*(b)* $RPSD_S$ *: the population SFP process will approach the perturbed equilibrium and taking the joint limit, we have*

$$\lim_{\rho \to 1} \lim_{t \to \infty} x(t) = \hat{p}$$

*players' mixed strategies will approach the logit equilibrium.*

**Proof:** These results follow from our earlier results on the behavior of the BR and PBR dynamics and the application of stochastic approximation theory. See Benaïm and Hirsch (1999), Hopkins (1999b, 2002), Hofbauer and Sandholm (2002) and Benaïm, Hofbauer and Hopkins (2009). □

## Appendix B (Experiment Instructions)

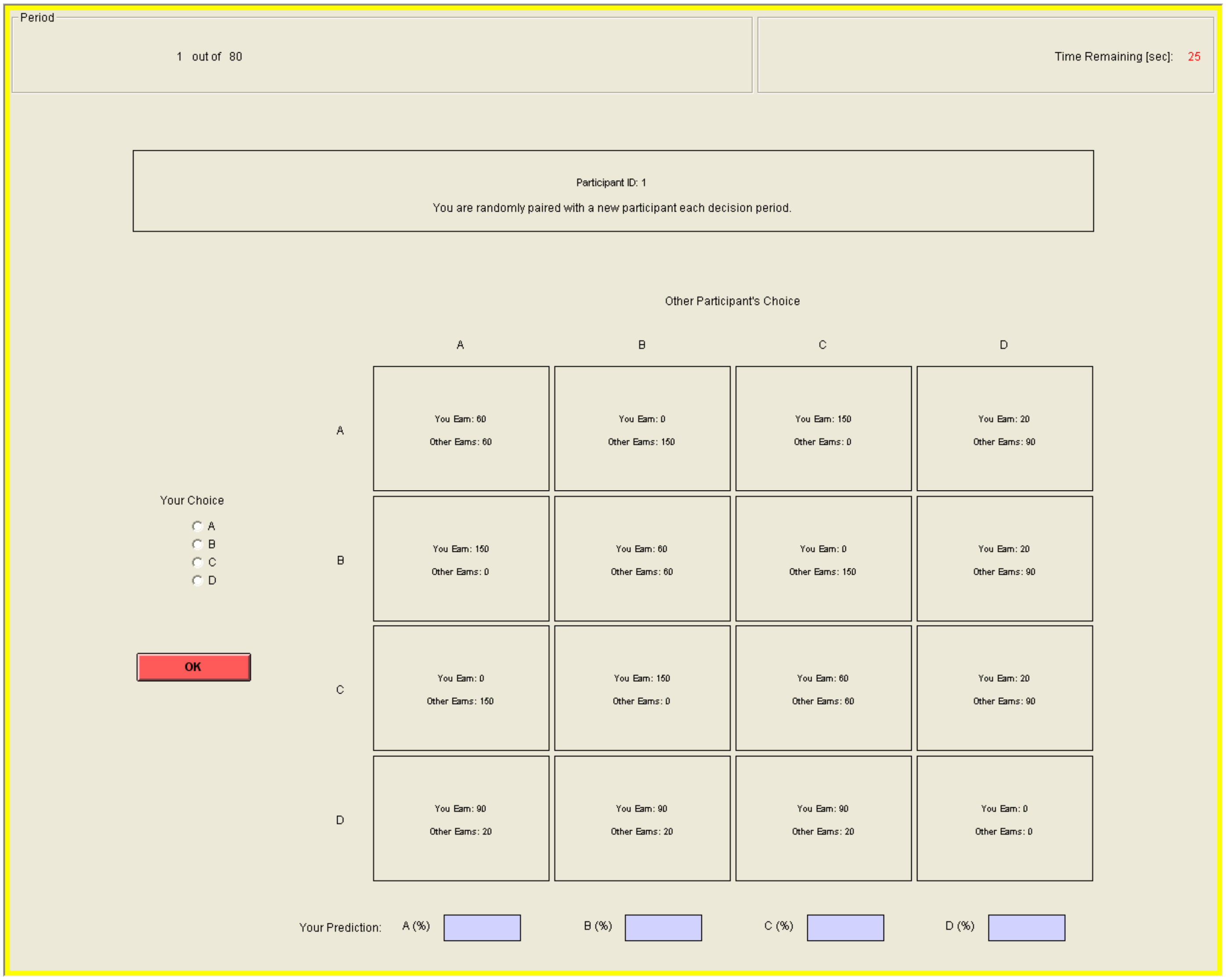
This is an experiment in the economics of strategic decision making. Various agencies have provided funds for this research. If you follow the instructions and make appropriate decisions, you can earn an appreciable amount of money. The currency used in the experiment is francs. Your francs will be converted to dollars at a rate of _____ dollars equals 100 francs. At the end of today's session, you will be paid in private and in cash for ten randomly-selected periods.

It is important that you remain silent and do not look at other people's work. If you have any questions, or need assistance of any kind, please raise your hand and an experimenter will come to you. If you talk, laugh, exclaim out loud, etc., you will be asked to leave and you will not be paid. We expect and appreciate your cooperation.

The experiment consists of 80 separate decision making periods. At the beginning of each decision making period you will be randomly re-paired with another participant. Hence, at the beginning of each decision making period, you will have a one in 11 chance of being matched with any one of the 12 other participants.

Each period, you and all other participants will choose an action, either A, B, C or D. An earnings table is provided on the decision screen that tells you the earnings you receive given the action you and your currently paired participant chose. See the decision screens on the next page. To make your decision you will use your mouse to click on the A, B, C or D buttons under *Your Choice:* and then click on the OK button.

Your earnings from the action choices each period are found in the box determined by your action and the action of the participant that you are paired with for the current decision making period. The values in the box determined by the intersection of the row and column chosen are the amounts of money (in experimental francs) that you and the other participant earn in the current period. These amounts will be converted to cash and paid at the end of the experiment if the current period is one of the ten periods that is randomly chosen for payment.




**Decision Screen**

To take a random example, if you choose **C** and the other participant chooses **D**, then as you can see in the square determined by the intersection of the third row (labeled C) and the fourth column (labeled D), you earn 20 francs and the other participant earns 90 francs. The 16 different boxes indicate the amounts earned for every different possible combination of A, B, C and D.

Predictions

When you make your action choice each period you will also enter your prediction about how likely the person you are paired with makes each of his or her action choices. In addition to your earnings from your action choices we will pay you an extra amount depending upon how good your prediction is.

To make this prediction you need to fill in the boxes to the right of *Your Prediction:* on the Decision Screen, indicating what the chances are that the participant you are paired with will make these choices. For example, suppose you think there is a 30% chance that this other person will choose C, and a 70% chance that he or she will choose D. This indicates that you believe that D is more than twice as likely as C, and that you do not believe that either A or B will be chosen. [The probability percentages must be whole numbers (no decimals) and sum to 100% or the computer won't accept them.]

At the end of the period, we will look at the choice actually made by the person you are paired with and compare his or her choice to your prediction. We will then pay you for your prediction as follows:

Suppose you predict that the person you are paired with will choose D with a 70% chance and C with a 30% chance (as in the example above), with 0% chances placed on A and B. Suppose further that this person actually chooses D. In that case your earnings from your prediction are

**Prediction Payoff** (D choice) = $5 - 5(0.7^2 + 0.3^2 + 0^2 + 0^2) + 10(0.70) = 9.1$ francs.

In other words, we will give you a fixed amount of 5 francs from which we will subtract and add different amounts. We subtract 5 times the sum of the squared probabilities you indicated for the four choices. Then we add 10 times the probability that you indicated for the choice of the person you are paired with actually made (0.7 probability in this example).

For these same example predictions, if the person you are paired with actually chooses A (which you predicted would happen with 0% probability), your prediction earnings are

**Prediction Payoff** (A choice) = $5 - 5(0.7^2 + 0.3^2 + 0^2 + 0^2) + 10(0) = 2.1$ francs.

Your prediction payoff is higher (9.1) in the first part of this example than in the second part of this example (2.1) because your prediction was more accurate in the first part.

Note that the lowest payoff occurs under this payoff procedure when you state that you believe that there is a 100% chance that a particular action is going to be taken when it turns out that another choice is made. In this case your prediction payoff would be 0, so you can never lose earnings from inaccurate predictions. The highest payoff occurs when you predict correctly and assign 100% to the choice that turns out to the actual choice made by the person you are paired with; in this case your prediction payoff would be 10 francs.

> **Note that since your prediction is made before you know which action is chosen by the person you are paired with, you maximize the expected size of your prediction payoff by simply stating your true beliefs about what you think this other person will do. Any other prediction will decrease the amount you can expect to earn from your prediction payoff.**

The End of the Period

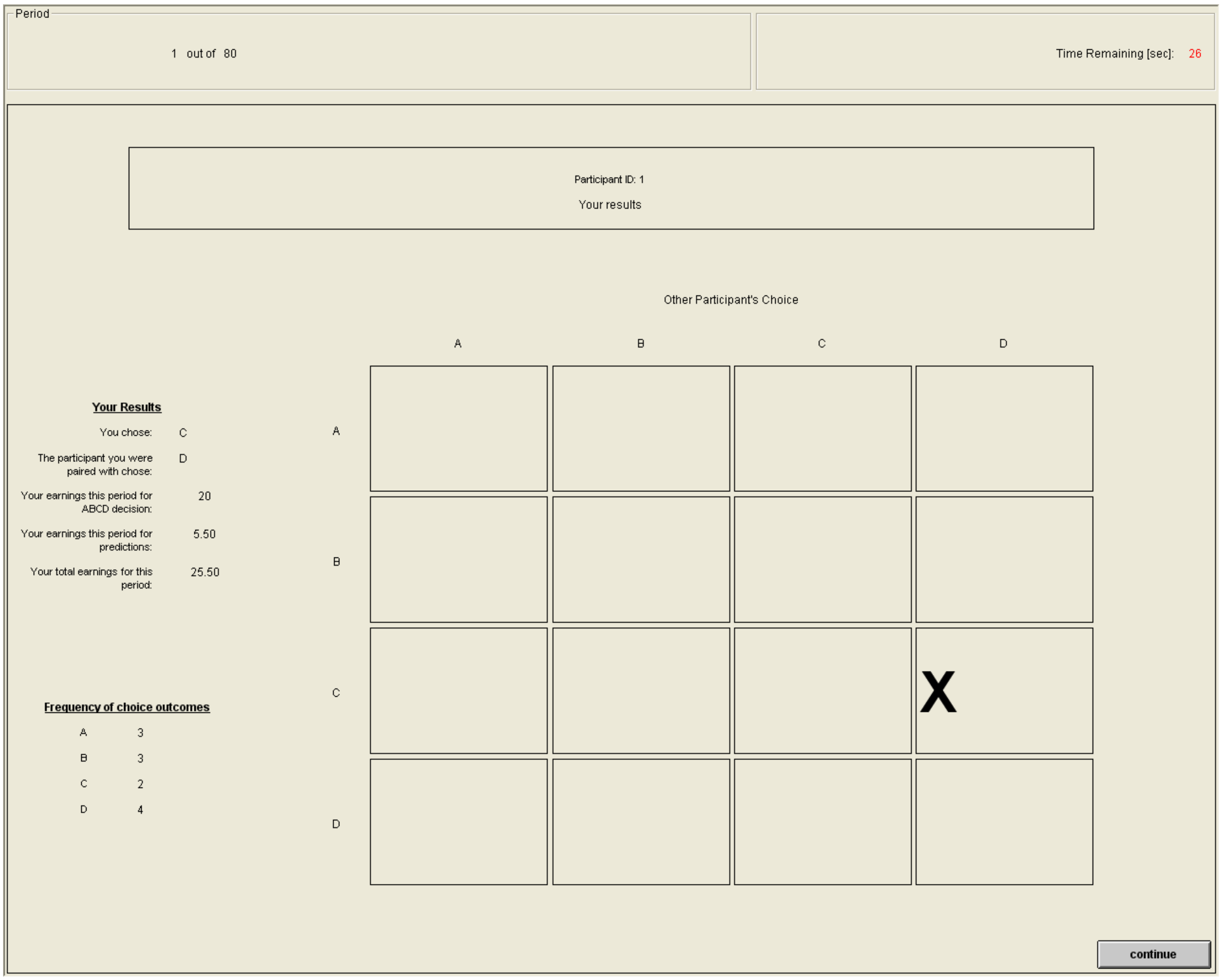
When all participants have made choices for the current period you will be automatically switched to the outcome screen, as shown on the next page. This screen displays your choice as well as the choice of the person you are paired with for the current decision making period. The chosen box is highlighted with a large **X**. It also shows your earnings for this period for your action choice (ABCD decision) and prediction, and your total earnings for the period. The outcome screen also displays the number of A, B, C and D choices made by all participants during the current period.

Once the outcome screen is displayed you should record your choice and the choice of the participant you were paired with on your Personal Record Sheet. Also record your earnings. Then click on the *continue* button on the lower right of your screen. Remember, at the start of the next period you are randomly re-paired with the other participants, and you are randomly re-paired each and every period of the experiment.

The End of the Experiment

At the end of the experiment we will randomly choose 10 of the 80 periods for actual payment using dice rolls (two ten-sided die, one with the tens digit and one with the ones digit). You will sum the total earnings for these 10 periods and convert them to a U.S. dollar payment, as shown on the last page of your record sheet.

We will now pass out a questionnaire to make sure that all participants understand how to read the earnings table and understand other important features of the instructions. Please fill it out now. Raise your hand when you are finished and we will collect it. If there are any mistakes on any questionnaire, I will go over the relevant part of the instructions again. Do not put your name on the questionnaire.




**Example Outcome Screen**

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