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Inequality Measures as Tests of Fairness in an Economy.

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Abstract

Standard measures of inequality have been criticized for a long time on the grounds that they are snap shot measures which do not take into account the process generating the observed distribution. Rather than focusing on outcomes, it is argued, we should be interested in whether the underlying process is "fair". Following this line of argument, this paper develops statistical tests for fairness within a well defined income distribution generating process and a well specified notion of "fairness". We find that standard test procedures, such as LR, LM and Wald, lead to test statistics which are closely related to standard measures of inequality. The answer to the "process versus outcomes" critique is thus not to stop calculating inequality measures, but to interpret their values differently—to compare them to critical values for a test of the null hypothesis of fairness.

1 Introduction

The standard procedure for measuring income inequality in a society is to take its observed distribution of income and to calculate an inequality index from it. However, this procedure is criticized on a number of grounds based on the fact that the observed distribution is nothing but a snap shot of the outcome of a process, and that it is the process that matters. The "snap shot" aspect of the critique is present in the burgeoning literature on the measurement of mobility, where the object is to evaluate distributions of inter temporal streams of income (see e.g.: Haider and Solon, 2006; Baker and Solon, 2003; Haider, 2001; Jenkins, 1987). While the mobility discussion raises interesting issues, it does not fully capture the "outcomes versus process" critique. This holds that it is the "fairness" of the underlying process that should be the basis for evaluation, not the outcomes.

The critique of an outcomes based focus is wide ranging, and is present in many strands in the literature. Sen (1973), for example, has criticized the "consequentialism" of Utilitarianism and, a fortiori, measures such as the Atkinson (1970) measure of inequality which are based on it. Nozick's (1974) theory of justice, sometimes seen as counter to that of Rawls (1971) also appeals to information on what brings about the outcomes we observe. As Atkinson and Bourguignon (2000) observe, "For him, it is not the distribution of income that matters but the process by which it is brought about, people being 'entitled' to resources that were justly acquired or that were transferred to them according to a just process, even if this means they will be immensely rich, and that their riches may be of no benefit to the poor."

Similar tensions arise in the context of Sen's Capability Approach (see e.g. Sen, 1985). As Sugden (1993) argues: "...... consider Joe who has a low standard of living in middle age because, when young, he chose not to continue his education beyond the minimum school-leaving age. In middle age, Joe's achieved functionings are less good than those of his twin brother Bill who chose to stay on at school. Joe's current capabilities are less rich too. But, viewed over their lives as wholes, both had similar capability sets. Does justice give Joe any claim on Bill's greater income?" (p. 1952)

A further related strand in the literature, which emphasizes process, is that on "equality of opportunity". As surveyed in Roemer (1996), these approaches try to identify those variations in income which are influenced by factors in the control of the individual (eg effort), and to devise operational means of correcting the observed inequality of incomes for these. An early proponent of this line of argument was Milton Friedman (1962), who brought in the consequences of risk taking for interpreting observed inequality:

"Another kind of inequality arising through the operation of the market is also required, in a somewhat more subtle sense, to produce inequality of treatment... It can be illustrated most simply by a lottery. Consider a group of individuals who initially have equal endowments and who agree voluntarily to enter a lottery with very unequal prizes. The resultant inequality is surely required to permit the individuals in question to make most of their initial equality... Much of the inequality of income produced by payment in accordance with product reflects 'equalizing' differences or the satisfaction of men's taste for uncertainty."

Friedman's contribution simultaneously highlights the issue of process versus outcomes, and the fact that even when the process implies ex ante equality (free lottery choice by identical individuals), the outcome may well show (misleadingly, in his view) inequality among individuals. Even though the process itself is "fair", inherent randomness may show spurious inequality in outcomes.

Suppose we wish to evaluate the process in Friedman's example, this would require an evaluation of whether the lotteries faced by the different individuals were indeed identical. If we could directly observe the lottery choices, that would be the end of the matter. But this is usually not the case. All we can observe are in fact the outcomes. The task is then to try and infer from these outcomes the nature of the process which generated them. It is clear that in order to do this we will have to provide a minimal structure to the class of processes. It is only within a given class of processes that we will be able to infer more specific properties of the process which gave rise to the outcomes we observe. But we hope to show that making these assumptions can provide considerable insight into the relationship between outcomes and process.

If we are only interested in process, does the standard procedure, of calculating inequality indices on the observed income distribution, have any place at all in the analysis? Surely they should be jettisoned? In this paper we assess the extent to which many conventional inequality measures may be interpreted as key statistics for testing the null hypothesis of "fairness" or "equity" in a process of income distribution generation. The analysis here suggests that the "process versus outcomes" critique could potentially be resolved by interpreting the numerical values of inequality indices differently - in particular, as statistics that help to determine whether the specified null hypothesis on the nature of the process can be rejected at a given level of significance.

The plan of the paper is as follows. Section 2 sets out the basic model and the family of processes we will be considering, and specifies what "fairness" or "equity" means in this context. Section 3 derives the Likelihood Ratio (LR), Lagrange Multiplier (LM) and Wald tests for fairness/equity, and shows the intimate links between these test statistics and standard measures of inequality. Section 4 extends the analysis, including more general approaches to hypothesis testing, which suggests families of inequality measures not generally encountered in the income distribution literature. Section 5 concludes.

2 The relationship between hypothesis tests of a null of equality and inequality indices.

Let y_i $(i = 1, 2..n)$ denote individual i's income, $Y = \sum_{i=1}^{n} y_i$ denote total income, $s_i = \frac{y_i}{Y}$ the income share of the ith individual and $\mu = \frac{Y}{n}$ be average income per head. Consider the process wherer each of the Y income units are allocated across the n individuals. Let the probability that individual i receives each unit of income be p_i. Each unit is assumed to be distributed independently so that the pdf of $y_1, y_2, ...y_n$ is multinomial with likelihood

$$L(y_1, y_2, ...y_{n-1}; p_1, p_2, ...p_{n-1}) = \frac{Y!}{y_1!y_2!...y_n!} p_1^{y_1} p_2^{y_2} ... p_n^{y_n} \qquad (2.1)$$

where $p_n = 1 - \sum_{i=1}^{n-1} p_i$ and $y_n = Y - \sum_{i=1}^{n-1} y_i$.

Throughout, we consider the null hypothesis that ex ante, each individual has an equal chance of receiving each unit of income. Hence under H_0 we have the n-1 restrictions

$$H_0: \qquad p_1 = p_2 = ... = p_{n-1} \qquad (2.2)$$

and an alterative H_1 that one or more of these restrictions are violated. This null hypothesis encapsulates precisely and in analytical terms what we mean by fairness/equity in the context of the current paper. We now examine the relationship between the LR, LM and Wald tests and commonly used inequality measures. We assume that n is fixed and Y is (asymptotically) large (see discussion below).

2.1 The LR test

The log likelihood (l) is

$$l = \ln(Y!) - \sum_{i=1}^{n} \ln y_i! + \sum_{i=1}^{n} y_i \ln p_i \tag{2.3}$$

A form for the LR test of (2.2) is found by comparing the values of (2.3) obtained when $\widehat{p_i} = \frac{1}{n}$ (the null "estimates") with that obtained under $\widehat{p_i} = \frac{y_i}{Y}$ (the alternative estimates). Hence we have

$$\begin{aligned} LR &= 2(l_1 - l_0) = 2\left(\sum_{i=1}^{n} y_i \ln \frac{y_i}{Y} - \sum_{i=1}^{n} y_i \ln \frac{1}{n}\right) && (2.4) \\ &= 2\left(\sum_{i=1}^{n} y_i(\ln s_i + \ln n)\right) = 2Y \sum_{i=1}^{n} s_i \ln n s_i && \\ &= 2Y.T && (2.5) \end{aligned}$$

where T is the Theil index.

2.2 The LM test

The score vector is

$$\frac{\partial l}{\partial p_i} = \frac{y_i}{p_i} - \frac{y_n}{p_n} \tag{2.6}$$

Evaluated under H_0 this becomes

$$g_0 = \frac{\partial l}{\partial p_i}|_{H_0} = n(y_i - y_n) \tag{2.7}$$

Differentiating (2.6) wrt $p_j, j = 1, 2..n-1$ gives the Hessian as

$$\frac{\partial^2 l}{\partial p \partial p'} = -diag\{\frac{y_i}{p_i^2}\} - \frac{y_n}{p_n^2}\mathbf{ii}' \tag{2.8}$$

where $\mathbf{i}$ is an $n-1x1$ vector of units. Taking expectations gives the (asymptotic) information matrix as

$$\mathbf{I}_{\infty} = -E\{\frac{\partial^2 l}{\partial p \partial p'}\} = nY\{\mathbf{ii}' + I_{n-1}\} \tag{2.9}$$

where I_{n-1} is the $n-1xn-1$ identity matrix and $\mathbf{I}_{\infty}$ is the asymptotic information matrix.

The LM test may be written as

$$LM = g_0' \mathbf{I}_{\infty}^{-1} g_0 \tag{2.10}$$

It is easy to show [1] that $\mathbf{I}_{\infty}^{-1}$ is

$$\mathbf{I}_{\infty}^{-1} = \frac{1}{nY}\{I_{n-1} - \frac{1}{n}\mathbf{ii}'\} \tag{2.11}$$

Using (2.11) and (2.7) in (2.10) gives

$$LM = \frac{n}{Y}\left(\frac{n-1}{n}\sum_{i=1}^{n-1}(y_i - y_n)^2 - \frac{2}{n}\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2}(y_i - y_n)(y_j - y_n)\right) \tag{2.12}$$

We may relate LM to two separate measures of inequality using the following tedious manipulations

$$\begin{aligned}
LM &= \frac{n}{Y}\left(\frac{n-1}{n}\sum_{i=1}^{n-1}(y_i - y_n)^2 - \frac{2}{n}\sum_{j=i+1}^{n-1}\sum_{i=1}^{n-2}(y_i - y_n)(y_j - y_n)\right) && (2.13)\\
&= \frac{n}{Y}\left(\sum_{i=1}^{n}(y_i - y_n)^2 - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n}(y_i - y_n)(y_j - y_n)\right) && (2.14)\\
&= \frac{n}{Y}\left(\sum_{i=1}^{n} y_i^2 + ny_n^2 - 2y_n\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} y_i y_j + ny_n^2 + 2y_n\sum_{i=1}^{n} y_i\right) && (2.15)\\
&= \frac{n}{Y}\left(\sum_{i=1}^{n} y_i^2 - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n} y_i y_j\right) = \frac{n}{Y}\left(\sum_{i=1}^{n} y_i^2 - n\mu^2\right) && (2.16)\\
&= \frac{n}{\mu} svar(y) = n\mu[cvar(y)]^2 && (2.17)
\end{aligned}$$

[1] Premultiplying I_{∞} by I_{∞}^{-1} manifestly gives the identity matrix.

where *svar* is sample variance (computed using n rather than $n-1$ as normalising factor) and *cvar* is the coefficient of variation. Using further manipulations, another form for the LM test is derived. Consider the inequality measure based on ordered income shares s_i^*

$$G^* = \sum_{j=1}^{n}\sum_{i=1}^{n}(s_i^* - s_j^*)^2 \tag{2.18}$$

This measure is closely related to the familiar Gini Coefficient. The main difference is that the Gini coefficient is proportional to $\sum_{j=1}^{n}\sum_{i=1}^{n}|s_i^* - s_j^*|$ whereas we have$\sum_{j=1}^{n}\sum_{i=1}^{n}(s_i^* - s_j^*)^2$.(see Kanbur, 1984 page 407)

Expanding the sum on the right gives

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{i=1}^{n}(s_i^* - s_j^*)^2 &= \sum_{j=1}^{n}\sum_{i=1}^{n}(s_i^* - s_j^*)^2 = n\sum_{j=1}^{n}{s_j^*}^2 + n\sum_{i=1}^{n}{s_i^*}^2 + 2\sum_{j=1}^{n}\sum_{i=1}^{n}s_i^* s_j^* \\
&= 2n\left(\sum_{i=1}^{n}s_i^2 - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n}s_i s_j\right) \\
&= \frac{2n}{Y^2}\left(\sum_{i=1}^{n}y_i^2 - n\mu^2\right) = 2\frac{LM}{Y}
\end{aligned}$$

Hence we have

$$LM = \frac{Y}{2}G^* \tag{2.19}$$

2.3 The Wald test

We compute the Wald test in two ways. First we take the null limit of the information matrix that is computed using estimates of the p_i under the alternative. In effect this amounts to using $\mathbf{I}_\infty$ in (2.9). We call this $W1$. Second we use the small sample estimate of $\mathbf{I}_\infty$ which employs estimates of p_i derived under the alternative. This, which we call $W2$, is the version of the Wald test that permeates empirical practice. $W1$ is not often used because it is rare that the information matrix is totally known under the null as it is here. For $W1$ we have

$$W1 = (\widehat{\mathbf{p}} - \mathbf{p}_0)'\mathbf{I}_\infty(\widehat{\mathbf{p}} - \mathbf{p}_0) \tag{2.20}$$

where $\widehat{\mathbf{p}}$ and $\mathbf{p}_0(= \frac{1}{n}\mathbf{i})$ are respectively $n-1x1$ vectors containing the estimated and null values of $p_i(i=1,..n-1)$. Again tedious manipulations yield an alternative more familiar formulation.

$$
\begin{aligned}
W1 &= (\widehat{\mathbf{p}}-\mathbf{p}_0)'\mathbf{I}_\infty(\widehat{\mathbf{p}}-\mathbf{p}_0) = \frac{1}{Y^2}(\mathbf{y}-\mu\mathbf{i})'nY\{\mathbf{ii}'+I_{n-1}\}(\mathbf{y}-\mu\mathbf{i}) \\
&= \frac{n}{Y}\left(\sum_{i=1}^{n-1}(y_i-\mu)^2+\left(\sum_{i=1}^{n-1}(y_i-\mu)\right)^2\right) \\
&= \frac{n}{Y}\left(\sum_{i=1}^{n}(y_i-\mu)^2-(y_n-\mu)^2+\left(\sum_{i=1}^{n}(y_i-\mu)\right)^2-(y_n-\mu)^2-2(y_n-\mu)\sum_{i=1}^{n-1}(y_i-\mu)\right)
\end{aligned}
$$

Now using the facts that $\sum_{i=1}^{n}(y_i-\mu)=0$ and $\sum_{i=1}^{n-1}(y_i-\mu)=y_n-\mu$ gives

$$
W1 = \frac{n}{Y}\left(\sum_{i=1}^{n}(y_i-\mu)^2\right) = LM \tag{2.21}
$$

Hence $W1$ reduces to LM. Perhaps of more interest therefore is $W2$. Here we use the small sample Hessian using estimates obtained under the alternative (denoted $\mathbf{I}_1$) under the justification that it consistently estimates $\mathbf{I}_\infty$. Using (2.8) and the estimates $p_i=\frac{y_i}{Y}$ gives

$$
\mathbf{I}_1 = Y^2\left(diag\{\frac{1}{y_i}\}+\frac{1}{y_n}\mathbf{ii}'\right) \tag{2.22}
$$

which in turn gives the Wald test as

$$
W2 = (\widehat{\mathbf{p}}-\mathbf{p}_0)'\mathbf{I}_1(\widehat{\mathbf{p}}-\mathbf{p}_0) = \frac{1}{Y^2}(\mathbf{y}-\mu\mathbf{i})'Y^2\{diag\{\frac{1}{y_i}\}+\frac{1}{y_n}\mathbf{ii}'\}(\mathbf{y}-\mu\mathbf{i}) \tag{2.23}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\frac{(y_i-\mu)^2}{y_i}+\frac{1}{y_n}\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(y_i-\mu)(y_j-\mu) \\
&= \sum_{i=1}^{n}\frac{(y_i-\mu)^2}{y_i}-\frac{(y_n-\mu)^2}{y_n}+\frac{1}{y_n}\left(\sum_{i=1}^{n-1}(y_i-\mu)\right)^2 \\
&= \sum_{i=1}^{n}\frac{(y_i-\mu)^2}{y_i}-\frac{(y_n-\mu)^2}{y_n}+\frac{1}{y_n}\left(\sum_{i=1}^{n}(y_i-\mu)+(y_n-\mu)\right)^2 \\
&= \sum_{i=1}^{n}\frac{(y_i-\mu)^2}{y_i} = Y\sum_{i=1}^{n}s_i\left(\frac{s_i-s^*}{s_i}\right)^2
\end{aligned} \tag{2.24}
$$

where $s^*[=\frac{1}{n}]$ in (2.24) is average income share. Using the fixed n large Y assumption we may expand $\ln\frac{s^*}{s_i}$ around unity as

$$\ln\frac{s^*}{s_i} = (\frac{s^*}{s_i} - 1) + \frac{1}{2Y}\left(\sqrt{Y}(\frac{s^*}{s_i} - 1)\right)^2 + \textit{smaller order terms}$$

Hence we may write

$$\frac{s_i - s^*}{s_i} = -\ln\frac{s^*}{s_i} + O_p(Y^{-1}) = \ln\frac{s_i}{s^*} + O_p(Y^{-1})$$

Using this in (2.24) gives an approximation for $W2$ as

$$\begin{aligned} W2 &= Y\sum_{i=1}^{n} s_i\left(\frac{s_i - s^*}{s_i}\right)^2 = Y\sum_{i=1}^{n} s_i\left(\ln\frac{s_i}{s^*} + O_p(Y^{-1})\right)^2 \\ &= Y\sum_{i=1}^{n} s_i\left(\ln ns_i\right)^2 \end{aligned} \tag{2.25}$$

(2.25) shows $W2$ to be related to the Theil index of inequality. Apart from the scaling factor Y, it differs from the Theil index in that it is the weighted average of the square of logs rather than the logs themselves.

2.4 Discussion

The above three tests, whilst interesting from the taxonomical perspective are of limited practical usage because they are scale dependent - measuring Y in cents rather than dollars multiplies the statistic by 100. Rescaling to eliminate (e.g.) the Y term in the numerator does not solve the problem because whilst the test statistic's numerical value would be independent of units of measurement its p-value would not. It is not reasonable to package "units of measurement" into the null to form a joint hypothesis. We would like to develop fair income generation processes that led to statistics that did not have this problem.

There are two developments that would eliminate scale problems. The first is where n grows large with Y but with μ fixed and the second where a ratio of statistics from two independent populations is considered.

2.5 Large Y and n with fixed μ

We consider a sequence of economies where Y grows large but where μ is fixed in each economy. The asymptote is different to above - n instead of Y . Allowing infinite n and therefore infinite numbers of parameters violates standard likelihood assumptions. We still focus on the analytical form given by the LR test but do not rely on standard likelihood asymptotic theory for its large sample distribution. We deal with the likelihood ratio statistic from each economy but extensions to the corresponding Wald and LM tests should be obvious. The "fair" income generation process we now envisage is slightly different to before. We now envisage a sequence of economies with increasing number of participants in them indexed by i where $i = (1, 2...\infty)$ is the number of individuals in economy i. Each economy has $i\mu$ separate units of income and again individual i has p_i probability of gaining one of these. Consider the sequence $Y_n(n = 1, 2, ...\infty)$ where $Y_n = n\mu$. Each economy is associated with the n random variables $y_{in} = \{y_{1n}, y_{2n}, ...y_{nn}\}$ and the Theil index/statistic

$$\tau_n = \sum_{i=1}^{n} s_{in} \ln n s_{in} \tag{2.26}$$

where $s_{in} = y_{in}/Y_n$. It is easy to show via a second order Taylor expansion for s_{in} around $\frac{1}{n}$ that

$$s_{in} \ln n s_{in} \approx \frac{n}{2}(s_{in} - \frac{1}{n})^2 = q_{in}$$

for large n. In turn, $nq_{in}(i = 1, ...n)$ are approximately independent $N(\frac{1}{2\mu}, \frac{1}{2\mu}^2)$ variates when n is large. Using a central limit theorem for triangular arrays such as that given in Billingsley(1981) we may establish that

$$\frac{2\mu}{\sqrt{n}} \sum_{i=1}^{n}(nq_{in} - \frac{1}{2\mu}) = \sqrt{n}(2\mu\tau_n - 1) = \sqrt{n}(\frac{LR_n}{n} - 1) \Rightarrow N(0, 1) \tag{2.27}$$

We see again that we can interpret a linear transformation of Theil's inequality measure as a test of a particular null of fairness. Unlike our earlier tests however this one is scale free in that it does not depend upon the units in which income is measured. Clearly the "fairness" null here is different to that tested above or rather the fair (null) process is different. Instead of "throwing" extra units of currency into an economy with a fixed population and testing equal chance of catching each unit we now throw extra individuals plus extra blocks of μ units of money into the economy and test equal probabilities of each individual receiving money.

2.6 Comparison of fairness in two populations

Typically in the inequality literature the investigator wishes to compare the income distribution of two populations, (often corresponding to regions or countries) using a sample of incomes from each. We could extend the above tests of fairness/equity to such a case. Explicitly we consider a null hypothesis that both populations exhibit fairness/equity against the alternative that one population is fair and the other unfair and vice versa. The proposed test would have no power to detect "equally" unfair income processes. The LR,LM and W tests above are distributed as χ^2_n variates. Denoting LR_i $(i = 1, 2)$ as the LR test from population i and assuming incomes in the two populations are independent then under a null that both population processes are "fair" we have

$$\frac{LR_1/n_1}{LR_2/n_2} = \frac{\mu_1}{\mu_2}\frac{T_1}{T_2} \Rightarrow F_{n_1 n_2} \tag{2.28}$$

where n_i, T_i are respectively the number of incomes and the Theil index obtained from economy i.

3 Tests of identical distributions

In this subsection we take a different tack in trying to establish the extent to which inequality measures may be interpreted as tests of fairness. Here we change the null hypothesis that individuals incomes have been drawn from the same pdf. The problem here is that without explicitly specifying the null pdf, this concept is vacuous. For example, one could specify a null pdf equal to the empirical pdf of the data, and then the null and alternatives would coincide. In effect, we may only test the null that the data were drawn from specific pdf's. This "fair" income generating process is a joint null - namely identical pdf's plus the particular pdf in hand.

We do not hide the fact that our choice of pdf's and test statistics is in part motivated by a desire to be able to interpret standard inequality measures as tests of fairness. A key element in this analyis then is the extent to which our test statistic and pdf choice are reasonable in the current context of a fair income generation process.We analyse two cases here :- a) the uniform pdf and the "Gap" test and b) the exponential.pdf's and the LR test.

3.1 a) Testing that the data were drawn from a common uniform pdf

Suppose that the null is that the y_i $(i = 1, 2..n)$ are *iid* $U[0, b]$. The uniform pdf is quite intuitive in terms of a priori fairness as it suggests that all incomes for all people are equally likely. A popular test of goodness of fit (alongside the Kolmogorov-Smirnoff and chi-square tests) is Moran's "gap" test.

Consider the ordered sequence of null df values $F_0(y_1), ...F_0(y_n)$, where F_0 is the null df and y_i is the *ith* smallest data point. Under the null, the difference between these points or "gaps" should be *iid* $U[0, 1]$ variates. Moran(1947) proposed the test

$$M = \sum_{i=1}^{n}(F_0(y_i) - F_0(y_{i-1}))^2 \tag{3.29}$$

where $F_0(y_0)$ is set to zero. The asymptotic distribution of M is similar but nonstandard. Critical values are however easily obtained by numerical simulation.

For the $U[0, b]$ pdf we have

$$M_U = \frac{1}{b^2}\sum_{i=1}^{n}(y_i - y_{i-1})^2 \tag{3.30}$$

Using 2μ as an estimator of b (we can use any consistent estimator and the MLE is to be avoided here) then we have an inequality measure based on the normalised (by mean income) square of neighbouring differences in incomes.

3.2 b) Testing that the data were drawn from a common exponential pdf

Suppose the null is that the y_i are *iid* $EXP(\beta)$.Whilst the exponential pdf is less intuitively reasonable from an a priori point of view it does have the ability to allow Theil's second inequality index as the likelihood ratio test of fair income generation as we now show.

Consider the LR test of a null that y_i has pdf $EXP(\beta)$ versus the alternative that y_i has pdf $EXP(\beta_i)$. Under the null we have

$$l_0 = -n \log \widehat{\beta_0} - \sum_{i=1}^{n} \frac{y_i}{\widehat{\beta_0}} \tag{3.31}$$

where $\widehat{\beta_0} = \mu$ is the mle of β under H_0. Under the alternative, we have

$$l_1 = -\sum_{i=1}^{n} \log \widehat{\beta_{i1}} - \sum_{i=1}^{n} \frac{y_i}{\widehat{\beta_{i1}}} \tag{3.32}$$

where $\widehat{\beta_{i1}}$ is the mle of β for individual i under the alternative. A quick glance at the form of l_1 shows that $\widehat{\beta_{i1}}$ is just y_i so that l_1 simplifies to

$$l_1 = -\sum_{i=1}^{n} \log y_i - n \tag{3.33}$$

Straightforwardly then the likelihood ratio test of equal means is proportional to the log of the average minus the average of the logs i.e.

$$LR = 2(l_1 - l_0) = 2n(\log \mu - \frac{\sum_{i=1}^{n} \log y_i}{n}) \tag{3.34}$$

This is just $2n$ times Theil's second inequality index and unlike the LR test in section 1.2 is scale free.

3.3 Conclusion

In this paper we have attempted to motivate the use of commonly discussed inequality indices in terms of tests of a null hypothesis consisting of a fair income generating *process*. We were only partially successful in our goals. Whilst we found that many commonly used indices could - taxonomically speaking - be interpreted as test statistics of a null of some "fair" income process or other, some of the tests considered led to scale dependent statistics which could not therefore be used in practical circumstances. Future work should expand the set of test statistics further beyond those motivated by the likelihood function. A further contribution would be to analyse the power of the tests presented to reject a false null of fairness.

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