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Matching and Sorting when Like Attracts Like*

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ABSTRACT: This paper examine a class of two-sided matching problems with non-transferable utility. Agents are horizontally differentiated, and each would prefer to be matched with a similar partner i.e. “like attracts like”. Such preferences imply a unique equilibrium assignment describing the pattern of matching; however, the pattern of assortment in equilibrium is found to depend critically on the distribution of types among the two sexes.

KEYWORDS: Matching; sorting; uniqueness; horizontal heterogeneity; marriage.

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1 INTRODUCTION

Do more educated men marry more educated women? Are more productive workers employed by larger firms? Are students from poor neighbourhoods taught by less qualified teachers? Such questions about the nature and degree of assortment displayed by two-sided matching markets are not only of interest in their own right but also have important implications for efficiency and welfare. It is not surprising, then, that they have received considerable attention.¹

In the case of fully transferable utility, a well established theoretical result is that regardless of the distribution of types matching will display positive assortment if the combined output of two matched agents is a supermodular function of their characteristics. If utility is partially but not fully transferable (i.e. if the utility possibility frontier for any pair of agents is negatively sloped but not linear) then, as Legros and Newman (2007) show, positive assortment can be guaranteed by a further condition on how the degree of transferability depends on the two agents' types.

If utility is not transferable at all, matters are very different. Agents can assess and rank potential partners without the need to take into account where they will be on a downward sloping possibility frontier. When considering beauty, or intelligence, or prestige, it might be reasonable to assume that all agents on one side of the market agree on how they rank the agents on the other side: everyone would prefer a more attractive spouse; all students would prefer to go to a more prestigious university; all universities would prefer to admit a cleverer student. With agents displaying such *vertical* heterogeneity, the most desirable agent on one side will match with the most desirable on the other side, the second most desirable agents will match with each other, and so on. Thus positive assortment will arise if there are no matching frictions and agents can freely choose with whom they match.

On the other hand, agents may be *horizontally* differentiated, and prefer to match with someone who is similar to them, or who fits in with their own objectives or capabilities; in brief, like may be attracted to like. For example, a woman might prefer a husband of the same age, or who has similar tastes or political views.² A student might prefer to go to a college where the courses are pitched at a level suitable to his or her ability; a college might prefer a less able student because it has a mission to educate the less gifted, not just the cleverest. A research oriented university may prefer, and be preferred, by an applicant who is dedicated to research and sees teaching as a necessary chore, whereas a university whose income depends mostly on teaching may prefer a brilliant teacher to the star researcher. Similarly, a hospital specialising in the development of new treatments for cancer may prefer an academically oriented intern who has done well in oncology, whereas the hospital with a vacancy in its busy city centre Accident and Emergency department would prefer someone with practical skills who can work under pressure. Still in the medical domain, doctors may differ in the way they prefer to treat a particular condition, and patients may differ in the treatment they would like to receive.³

With preferences such as these, it is not at all clear that positive sorting will be the outcome, however desirable. For a hospital wanting to hire a new intern, there may be no graduating medical student with the exact mix of abilities and character that it is looking for; and its preferred candidate, of those available, may prefer to work at another hospital. Similarly, for a prospective intern there may be no vacancy in her ideal hospital. Who then gets matched with

¹The literature is huge; for a small sample that nevertheless conveys something of the range of applications, see Andrews, Schank and Upward (2007), Blanden (2005), Boyd et al. (2003), Fabbri and Monfardini (2006), Fernandez, (2002), Phelps (2000)

²On preferences for similarity of age see Hayes (1995), for similarity of weight see Schafer and Keith (1990), for similarity of other traits see Buss and Barnes (1986). For an economic model of preferences for a partner with similar tastes for jointly consumed goods, see Clark and Kanbur (2004).

³This is explicitly seen as a matching issue in Phelps (2000); in Fabbri and Monfardini (2006) this approach is used to analyse modes of childbirth.

whom is of course the very stuff of matching theory.⁴ As an example, consider a marriage market in which men and women are characterised only by height and each person prefers a partner closer to them in height. Suppose there are two women, Cherie and Laura, of height 1.50m and 1.65m respectively; and two men, George and Tony, of height 1.60m and 1.75m. Then Laura and George prefer each other and must be matched in equilibrium, leaving Cherie (the shorter woman) and Tony (the taller man) to be matched. Thus we have negative assortment.

The outcome in this particular case depends both on the non-transferability of utility and on the distribution of height among the four agents. If a matched couple jointly produced a divisible (i.e. transferable) output then in equilibrium total output must be maximised; if the couple’s joint output is a supermodular function of their characteristics (such as the product of their heights) this requires positive assortment, so Cherie would be matched with George and Laura with Tony. But once we move away from fully transferable utility, “for the analyst seeking to characterize the equilibrium matching patterns in such settings, there is little theoretical guidance” (Legros and Newman, 2007). To illustrate how the matching pattern might change in the example above, if George were 1.55m tall, then he and Cherie would prefer each other and we would have positive assortment.

This paper goes some way to fill the gap identified by Legros and Newman. I analyse matching and sorting when utility is non-transferable and like attracts like. I look for conditions under which we get complete positive or negative assortment and I also analyse what determines the degree of assortment in intermediate cases. The plan of the remainder of the paper is as follows: in the next section I set up a formal model of two sided matching within which matching, stability, and assortment can be defined and analysed. Section 3 show the existence and uniqueness of the equilibrium matching pattern. Section 4 is less formal and draws out the implications of the previous analysis for our understanding of matching and sorting in a range of simple cases. Section 5 discusses two simple generalisations of the basic model and Section 6 briefly concludes.

2 THE MODEL

2.1 The distribution of types

I study an atomless economy in which agents are characterised by sex and by type. An agent’s type is a real number in the interval $T = [\underline{t}, \bar{t}]$. I allow for a continuum of agents of any given type, but no mass points of men or women in T . I therefore take as primitives functions $f : T \rightarrow R$ and $g : T \rightarrow R$ that are nonnegative, bounded, and integrable. f and g give the density of types amongst men and women respectively.⁵

The masses of men and women with types in a subset A of T are denoted by $\lambda_m(A)$ and $\lambda_w(A)$ respectively,⁶ where

$$\begin{aligned}\lambda_m(A) &= \int_A f(x)dx \\ \lambda_w(A) &= \int_A g(x)dx.\end{aligned}$$

$\lambda_m([\underline{t}, t])$ and $\lambda_w([\underline{t}, t])$ are continuous functions of t , both from T to R , denoted by $F(t)$ and $G(t)$ respectively. I assume $F(\bar{t}) = G(\bar{t}) = 1$.⁷

⁴Roth has analysed the intern market extensively. See for example, Roth (1984)

⁵Throughout the paper, “integrable” means Lebesgue integrable and “measurable” means Lebesgue measurable.

⁶Formally, λ_m and λ_w are measures on the measure space (T, L_T) , where L_T is the set of measurable subsets of T .

⁷Thus the masses of men and women are equal; and we can think of mass as proportion. The consequences of

The density functions f and g have the same domain, T , but their supports, denoted T_f and T_g , generally differ. I assume that T_f and T_g are intervals $[\underline{t}_f, \bar{t}_f]$ and $[\underline{t}_g, \bar{t}_g]$ respectively, that $f(t) > 0$ if and only if $t \in \text{int}(T_f)$, and $g(t) > 0$ if and only if $t \in \text{int}(T_g)$, and that f and g are continuous at all points in $\text{int}(T_f)$ and $\text{int}(T_g)$ respectively. This rules out situations where, for example, there are short and tall men, but none in an intermediate range (although such a situation could be approximated by a bimodal distribution). It follows that F is strictly increasing over the interval T_f with inverse $F^{-1} : [0, 1] \rightarrow T_f$, and G is strictly increasing over the interval T_g with inverse $G^{-1} : [0, 1] \rightarrow T_g$.⁸

A central role in determining matching patterns is played by the functions $h = \min(f, g)$ and $k = f - g$. Let $T_h = T_f \cap T_g$; then $h(t) > 0$ only if $t \in T_h$. Of course, if all of one sex are taller than all those of the other, then T_h is empty. As for k , I make the following assumption, which ensures that unless $f = g$ the function k crosses the zero axis only a finite number, $n - 1$, of times.

Assumption 1. If $f \neq g$, then there exists a finite number $n > 1$ of intervals $T_1 = [t_0, t_1], T_2 = [t_1, t_2], T_3 = [t_2, t_3], \dots, T_n = [t_{n-1}, t_n]$, where $t_0 = \underline{t}$ and $t_n = \bar{t}$, all of positive length, such that (a) either (i) for all $t \in T_i$, $k(t) \geq 0$ if i is odd and $k(t) \leq 0$ if i is even; or (ii) for all $t \in T_i$, $k(t) \leq 0$ if i is odd and $k(t) \geq 0$ if i is even; and (b) for $i = 1, \dots, n$, $\int_{T_i} k(x) dx \neq 0$.

2.1.1 Men, women, and subpopulations

The set of men is given by $M = \{s, x | s \in T_f, 0 \leq x \leq f(s)\}$. Each man is therefore an ordered pair (s, x) , where s denotes his type and x uniquely identifies his position in the continuum of men with type s . Similarly, the set of women is $W = \{t, y | t \in T_g, 0 \leq y \leq g(t)\}$. Since men and women are elements of R^2 , to measure sets of men and women we use λ_2 , the Lebesgue measure on R^2 . By construction, if $S = \{(s, x) \in M | s \in A\}$ then $\lambda_2(S) = \lambda_m(A)$, and if $S = \{(t, y) \in W | t \in A\}$, then $\lambda_2(S) = \lambda_w(A)$. I refer to $P = (M, W)$ as *the population*.

A *subpopulation* of $P = (M, W)$ is a pair $P' = (M', W')$, such that $M' \subseteq M$, $W' \subseteq W$, and $\lambda_2(M') = \lambda_2(W')$, where M' and W' are measurable subsets of R^2 ; for convenience I write $P' \subseteq P$. Similarly, a subpopulation $P'' = (M'', W'')$ of P is also a subpopulation of P' if $M'' \subseteq M'$ and $W'' \subseteq W'$; I write $P'' \subseteq P'$. If $M''' = M' \setminus M''$ and $W''' = W' \setminus W''$ then $P''' = (M''', W''')$ is also a subpopulation of P' and I write $P''' = P' \setminus P''$.

I measure the mass of a subpopulation $P' = (M', W')$ by the mass of M' , or equivalently that of W' , and by a slight abuse of notation I denote it by $\lambda_2(P')$; thus $\lambda_2(P') = \lambda_2(M') = \lambda_2(W')$.

2.2 Preferences

I assume “like attracts like”: given a choice between two women, a man prefers the woman whose type is closer to his own. Ideally he would like a partner with exactly the same type as his own. Similarly, each woman prefers a man whose type is closer to her own. Thus a sufficient statistic for an agent to assess a potential partner is the absolute difference in their types. Formally, a man (s, x) strictly prefers a woman (t, y) to a woman (t', y') if $|s - t| < |s - t'|$; similarly, a woman (t, y) strictly prefers a man (s, x) to a man (s', x') if $|s - t| < |s' - t|$. Furthermore, I assume that

dropping this assumption that are examined in Section 5.

⁸The assumption that f and g are zero at the endpoints of T_f and T_g respectively is only relevant if there is a discontinuity in f or g at an endpoint. The convention I adopt is analytically convenient and has no effect on the measures λ_m and λ_w or the distribution functions F and G .

for all agents any match with someone of the opposite sex, whatever the difference in types, is preferred to no match.⁹

2.3 Matchings

Since all agents would prefer to be paired with any member of the opposite sex than to remain single, and as $\lambda_m(T) = \lambda_w(T)$, we focus on outcomes where every agent is matched with someone of the opposite sex. We therefore define a matching of P to be a measurable bijection $\mu : M \rightarrow W$, with inverse μ^{-1} which is measure consistent i.e. $\lambda_2(S) = \lambda_2(\mu(S))$ for any measurable subset S of M , where $\mu(S) = \{w \in W : w = \mu(m) \text{ for some } m \in S\}$. If $w = (t, y), m = (s, x)$, and $w = \mu(m)$, then it is sometimes convenient to write $t = \mu_1(s, x), y = \mu_2(s, x), s = \mu_1^{-1}(t, y)$, and $x = \mu_2^{-1}(t, y)$. A matching of a subpopulation $P' = (M', W')$ is a measurable bijection $\mu' : M' \rightarrow W'$, with inverse μ'^{-1} , such that $\lambda_2(S) = \lambda_2(\mu'(S))$ for any measurable $S \subseteq M'$.

2.4 Matching as assignment of types

I denote by L_{T^2} the set of measurable subsets of T^2 , where I adopt the convention that if $(s, t) \in T^2$ then s refers to a man and t to a woman. Any matching μ induces a measure on the measurable space (T^2, L_{T^2}) which we can interpret as a joint distribution of types, enabling us to answer the question: what types of men are matched by μ with what types of women? Such a measure I label an *assignment*. For any matching μ , let θ_μ denote the assignment defined by¹⁰

$$\theta_\mu(C) = \lambda_2(\{(s, x) \in M : (s, \mu_1(s, x)) \in C\}).$$

$\theta_\mu(A \times B)$ denotes the mass of pairs such that the man's type is in A and the woman's in B . Obviously, for any matching $\mu, \theta_\mu(A \times T) = \lambda_m(A), \theta_\mu(T \times B) = \lambda_w(B)$ and $\theta_\mu(T \times T) = 1$.¹¹ However, the argument of θ_μ can be any measurable subset of T^2 , not just a rectangle of the form $A \times B$. For example, let $D = \{(t, t') \in T^2 \mid t = t'\}$. This is the 45° line in T^2 , representing pairs where the man and woman have the same type. Then $\theta_\mu(D)$ measures that part of the population where like is matched with like.

If μ' is a matching of a subpopulation P' , then it also induces a measure on (T^2, L_{T^2}) , denoted by $\theta_{\mu'}$, defined as above (with μ' in place of μ), and with the property that $\theta_{\mu'}(T \times T) = \lambda_2(P')$.

2.5 Assortative Matching

We can use the assignment θ_μ to analyse the type of assortment (if any) exhibited by the matching μ . Let $\alpha : T_f \rightarrow T_g$ be the strictly increasing bijection defined by $\alpha(s) = G^{-1}(F(s))$, and let θ^α be the measure on (T^2, L_{T^2}) defined by

$$\theta^\alpha(C) = \int_{\{s : (s, \alpha(s)) \in C\}} f(s) ds.$$

Thus $\theta^\alpha(C) = \theta^\alpha(C \cap \Gamma_\alpha)$ and $\theta^\alpha(\Gamma_\alpha) = 1$, where Γ_α is the graph of α . Then we shall say that a matching μ exhibits *strictly positive assortment* (SPA) if $\theta_\mu = \theta^\alpha$, i.e. if all pairs, except possibly for a set of measure zero, lie on Γ_α . Similarly, let $\beta : T_f \rightarrow T_g$ be the strictly decreasing bijection defined by $\beta(s) = G^{-1}(1 - F(s))$, and let θ^β be the measure on (T^2, L_{T^2}) defined by

⁹This assumption is dropped in Section 5.

¹⁰Note that we measure the mass of a set of pairs by the mass of paired men. Since there are no same sex pairings, we could have used the mass of paired women.

¹¹(i) and (ii) are similar to definition. of θ as a matching, as in Gretsky et.al. 1992.

$$\theta^\beta(C) = \int_{\{s:(s,\beta(s)) \in C\}} f(s) ds.$$

Then we shall say that a matching μ exhibits *strictly negative assortment* (SNA) if $\theta_\mu = \theta^\beta$.

2.5.1 How to derive assortative matchings of P

Matchings that exhibit SPA can be achieved in many ways. Let the matching $\hat{\mu}$ be defined by

$$\begin{aligned} \hat{\mu}_1(s, x) &= \alpha(s) \\ \hat{\mu}_2(s, x) &= \begin{cases} 0 & \text{if } x = \underline{t}_f \\ \frac{xg(\alpha(s))}{f(s)} & \text{if } \underline{t}_f < x < \bar{t}_f \\ 0 & \text{if } x = \bar{t}_f \end{cases} \end{aligned} \quad (1)$$

This matches every s type man with an $\alpha(s)$ type woman, and although there is no reason to expect that $f(s) = g(\alpha(s))$, nevertheless $\hat{\mu}$ is measure consistent. To see this, consider the rectangle $S = [s, s'] \times [x, x']$; the mass of women matched by $\hat{\mu}$ with the men in S is $\lambda_2(\hat{\mu}(S)) = \int_{\alpha(s)}^{\alpha(s')} (x' - x) \frac{g(\alpha(\sigma))}{f(\sigma)} d\sigma$. But α is differentiable with derivative $\frac{f(s)}{g(\alpha(s))}$, so $\lambda_2(\hat{\mu}(S)) = (x' - x) \int_{\alpha(s)}^{\alpha(s')} \frac{1}{\alpha'(\sigma)} d\sigma = (x' - x)(s' - s) = \lambda_2(S)$. This result extends to any countable union of rectangles and thence, from the definition of Lebesgue measure, to any measurable set. By a similar argument, the matching $\tilde{\mu}$ defined by

$$\begin{aligned} \tilde{\mu}_1(s, x) &= \beta(s) \\ \tilde{\mu}_2(s, x) &= \begin{cases} 0 & \text{if } x = \underline{t}_f \\ \frac{xg(\beta(s))}{f(s)} & \text{if } \underline{t}_f < x < \bar{t}_f \\ 0 & \text{if } x = \bar{t}_f \end{cases} \end{aligned} \quad (2)$$

exhibits SNA and is measure consistent.¹²

2.6 Stable matchings

A pair $(m, w) \in M \times W$ can *block* the matching μ if $w \neq \mu(m)$, m strictly prefers w to $\mu(m)$, and w strictly prefers m to $\mu^{-1}(w)$. In finite economies, the solution concept typically used in matching problems is the stable matching: a matching is stable if no pair can block it. In atomless economies the requirement that no pair can block a stable matching is excessive and rules out as unstable those matchings that can be blocked but only by a set of agents of measure zero. Indeed, it is possible to have a matching μ such that almost every agent in P can form part of pair that blocks μ , yet by taking away two men m_1 and m_2 and two women w_1 and w_2 to form $P' = (M \setminus (m_1 \cup m_2), W \setminus (w_1 \cup w_2)) = (M', W')$ then the resulting matching μ' of P' (i.e. $\mu'(m) = \mu(m)$ for $m \in M'$) cannot be blocked by *any* pair $(m, w) \in M' \times W'$.¹³ This motivates the following definitions:

¹²The reasoning used to show measure consistency would break down if α were a non-decreasing correspondence, i.e. if (in obvious notation) $s < s' \rightarrow t \leq t'$ for any $t \in \alpha(s)$ and $t' \in \alpha(s')$, rather than $s < s' \rightarrow t < t'$. Then measure consistency would fail wherever the graph of α was flat or vertical; for example if $\alpha(s) = \{t\}$ for any $s \in [s, s']$, where $s < s'$, then for $M' = \{(a, b) \in M | a \in [s, s']\}$, $\lambda_2(M') = \lambda_m([s, s']) > 0$, whereas $\lambda_2(\mu(M')) = \lambda_2(t) = 0$. It follows that there is no loss of generality in considering strictly and not weakly positive assortment, as the latter implies the former. Similarly, weakly negative assortment implies strictly negative assortment.

¹³For example, suppose $M = [0, 1] \times [0, 1]$ and $W = [1, 2] \times [0, 1]$ (so men's and women's types are distributed uniformly on $[0, 1]$ and $[1, 2]$ respectively) and let $m_1 = (0, 0)$, $m_2 = (1, 0)$, $w_1 = (1, 0)$, $w_2 = (2, 0)$. For $0 \leq s \leq 1$, let μ match man (s, x) with woman $(2 - s, x)$, except that $\mu(m_1) = (w_1)$ and $\mu(m_2) = (w_2)$. Then μ can be blocked by (m_2, w) for any $w = (t, y) \in W$ such that $t > 1$, and also by (m, w_1) for any $m = (s, x) \in M$ such that $s < 1$. But μ' cannot be blocked by any pair.

Definition 1 1 A matching μ of $P = (M, W)$ is totally stable if it cannot be blocked by any pair $(m, w) \in M \times W$.

Definition 2 2 A matching μ of $P = (M, W)$ is stable if P has a subpopulation $P' = (M', W')$ such that $\lambda_2(P') = 1$, $\mu(M') = W'$, and the matching μ' is a totally stable matching of P' , where $\mu'(m) = \mu(m)$ for $m \in M'$.

If μ is a stable matching of P , then θ_μ is called a stable assignment of P . Note that the matchings μ and μ' in Definition 2 satisfy $\theta_\mu = \theta_{\mu'}$ where $\theta_{\mu'}(C) = \lambda_2(\{(s, x) \in M' : (s, \mu'_1(s, x)) \in C\})$

3 STABILITY AND THE DISTRIBUTION OF TYPES

We begin by looking at two important special cases.

3.1 Case (i): $F = G$

If $F = G$, then it is possible for almost everyone to have an ideal match.

Theorem 1 If $F = G$, the matching $\hat{\mu}$, defined in equation (1), is a stable matching of P .

Note that if $f = g$ everywhere, $\hat{\mu}$ is the identity mapping as then $M = W$. There will in general be an uncountable number of ways for the set of type t men to match with the set of type t women, and so given the continuum T of types, there is an uncountable number of matchings that give rise to the same measure. But because any off-diagonal pairings create blocking pairs, it is only these matchings of P that are stable. Formally:

Theorem 2 If $F = G$, then a matching μ of P is stable if and only if $\theta_\mu = \theta^\alpha$.

This implies that although $\hat{\mu}$ may not be the unique stable matching of P if $F = G$, θ^α is the unique stable assignment, and any stable matching exhibits strictly positive assortment. In contrast:

Theorem 3 If $F \neq G$, then a matching μ of P is stable only if $\theta_\mu \neq \theta^\alpha$.

If positive assortment occurs only if $F = G$, then we must conclude that it is a highly atypical outcome of the kind of matching markets analysed here. If we do not have positive assortment, what is the matching pattern? The remainder of the paper can be seen as an answer to this question.

3.2 Case (ii): for some $t^* \in T$, either $F(t^*) = 0$ and $G(t^*) = 1$ or $F(t^*) = 1$ and $G(t^*) = 0$.

Without loss of generality, we consider the first possibility: $F(t^*) = 0$ and $G(t^*) = 1$; this formalises the condition that no man is shorter than any woman. In this case, all women prefer the shorter of any two men in M and all men prefer the taller of any two women in W . In a finite

population, this would imply that the shortest man would match with the tallest woman (since they prefer each other), the second shortest man would match with the second tallest woman (since they prefer each other from the remaining population), and so on, leading to a stable matching with negative sorting. In the present case, a similar outcome occurs.

Theorem 4 *If there exists some $t^* \in T$ such that $F(t^*) = 0$ and $G(t^*) = 1$, then (i) $\tilde{\mu}$, defined in equation (2), is a stable matching of P ; and (ii) $\tilde{\mu}$ is a stable matching of P if and only if $\theta_\mu = \theta^\beta$.*

The functions F and G can be perturbed and still satisfy $F(t^*) = 0$ and $G(t^*) = 1$. In this sense, negative assortment is a more robust outcome than positive assortment. Nevertheless, both cases (i) and (ii) might be considered as somewhat special, and we now consider the more general case where there is some but not complete overlap between f and g . We begin by establishing some general propositions regarding matching in such situations.

3.3 Matching in overlapping subpopulations

Recall that $h = \min(f, g)$, and let $\int_{T_h} h(x)dx = \Omega$. Thus Ω measures the degree of overlap between the two densities f and g . Cases (i) and (ii) above may be considered as special cases with Ω equal to 1 and 0 respectively. We shall say that $P' = (M', W')$ is an *overlapping* subpopulation (or more concisely an *overlap*) of P if $\lambda_2(P') = \Omega$ and the distribution of types both in M' and in W' is given almost everywhere by h , i.e. if $\lambda_2(\{(s, x) \in M' | s \in A\}) = \lambda_2(\{(t, y) \in W' | t \in A\}) = \int_A h(x)dx$. An important example arises when $M^0 = W^0 = M \cap W$; then $P^0 = (M^0, W^0)$ is an overlap of P .

Much of the analysis of P when $F = G$ can be carried over to overlaps of P . Importantly, almost all pairs in a stable matching of an overlap must have types that lie on the diagonal of $T \times T$, so a stably matched overlap exhibits strictly positive assortment. We state without proof:

Lemma 1 *(i) The mapping $\iota^0 : M^0 \rightarrow W^0$ defined by $\iota^0(m) = m$ is a stable matching of P^0 ; (ii) if $P' = (M', W')$ is an overlap of P , then $\mu' : M' \rightarrow W'$ is a stable matching of P' if and only if $\theta_{\mu'} = \theta_{\iota^0}$.*

It follows from part (ii) of this Lemma that μ' is a stable matching of P' if and only if the mass of pairs whose types lie on the diagonal $D, \lambda_2(\{(s, x) \in M' : (s, \mu'_1(s, x)) \in D\})$, equals Ω . The significance of this is that if P has some overlap between f and g , then any stable matching of P must match like with like to the maximum extent possible. Formally

Lemma 2 *If μ is a stable matching of P then $\theta_\mu(D) = \Omega$.*

It now follows that a stable matching μ of P must consist of a stable matching μ' of an overlapping subpopulation P' and a stable matching μ'' of the remaining subpopulation $P'' = P \setminus P'$ (the complement of P'). Formally,

Theorem 5 *μ is a stable matching of $P = (M, W)$ if and only if P has an overlapping subpopulation $P^1 = (M^1, W^1)$ such that (a) μ' is a stable matching of P^1 , where $\mu'(m) = \mu(m)$ for $m \in M^1$ and (b) μ^2 is a stable matching of $P^2 = (P \setminus P^1)$, where $\mu^2(m) = \mu(m)$ for $m \in M \setminus M^1$.*

The analysis of matching and sorting patterns when P has some overlap uses this proposition as its starting point. To find a stable matching of P , we must first identify an overlap P^1 and a

stable matching of P^1 . P^0 and ι^0 satisfy this requirement, but by part (ii) of Lemma 1 all stable matchings of all overlaps of P have the same measure and exhibit positive sorting, with almost all matched pairs having types on the diagonal D . The remaining population, $P^2 = (M^2, W^2)$, by definition, has no overlap, and therefore there is no further possibility of matching like with like. The precise pattern of matching of the subpopulation P^2 depends on the distribution of types amongst M^2 and W^2 , given by $F^2 = F - H$ and $G^2 = G - H$ respectively, where $H(t) = \int_t^1 h(x)dx$. Note that $F^2(\bar{t}) = G^2(\bar{t}) = 1 - \Omega$.

The next step is to identify subpopulations of P^2 that have stable matchings that can form part of an overall stable matching of P^2 and hence of P . Formally,

Definition 3 *If a subpopulation $\widehat{P} = (\widehat{M}, \widehat{W})$ has a subpopulation $P^\# = (M^\#, W^\#)$ and $\mu^\#$ is a stable matching of $P^\#$ such that for any $m^\# \in M^\#$ and any $\widehat{w} \in \widehat{W} \setminus W^\#$, $m^\#$ prefers $\mu^\#(m^\#)$ to \widehat{w} and for any $w^\# \in W^\#$ and any $\widehat{m} \in \widehat{M} \setminus M^\#$, $w^\#$ prefers $\mu^{\#-1}(w^\#)$ to \widehat{m} then $\mu^\#$ is a fixed matching of $P^\#$ relative to \widehat{P} .*

Then the following are almost immediate:

Lemma 3 *If $P^\# = (M^\#, W^\#)$ and $P^\sim = (M^\sim, W^\sim)$ are subpopulations of $\widehat{P} = (\widehat{M}, \widehat{W})$ with no agents in common, and $\mu^\#$ is a fixed matching of $P^\#$ relative to \widehat{P} and μ^\sim is a fixed matching of P^\sim relative to \widehat{P} , then μ^* is a fixed matching of P^* relative to \widehat{P} , where $P^* = (M^\# \cup M^\sim, W^\# \cup W^\sim)$ and $\mu^*(m) = \mu^\#(m)$ for $m \in M^\#$ and $\mu^*(m) = \mu^\sim(m)$ for $m \in M^\sim$.*

Lemma 4 *If $P^\# = (M^\#, W^\#)$ is a subpopulation of $\widehat{P} = (\widehat{M}, \widehat{W})$, $P^\sim = \widehat{P} \setminus P^\#$, μ is a matching of \widehat{P} where $\mu(m) = \mu^\#(m)$ for $m \in M^\#$ and $\mu^\sim(m) = \mu(m)$ for $m \in \widehat{M} \setminus M^\#$, then μ is stable if $\mu^\#$ is a fixed matching of $P^\#$ relative to \widehat{P} and μ^\sim is stable.*

We can now find a stable matching of P^2 as follows: find a fixed matching μ^3 of $P^3 \subseteq P^2$ relative to P^2 , (possibly made up of separate fixed matchings in accordance with the requirements of Lemma 3), form $P^4 = P^2 \setminus P^3$, find a fixed matching of $P^5 \subseteq P^4$, relative to P^4 and so on. Carrying on in this way, if the sequence of fixed matchings eventually comes to an end, we will have matched the entire population P . Such a sequence of fixed matchings does indeed exist. Thus

Theorem 6 *A stable matching of P exists.*

The proof is somewhat involved, but the intuition behind it is straightforward.¹⁴ First we match the overlap P^0 with ι^0 . Then a revealing way to characterise $P^2 = P \setminus P^0$ is to graph the function $k = f - g$. Above the horizontal axis, k gives the density of male types, f^2 ; below it gives the density of female types, g^2 . Figure 1 illustrates. Then where k crosses the axis we can find pairs of men and women with types very close to each other. Starting from, say, the point t_1 we can construct a subpopulation of men shorter than t_1 and women taller than t_1 and sort them negatively. In doing this we form a fixed matching of a subpopulation of P^2 , with the types of all agents in the subpopulation being within a distance d of each other, and the shortest man and the tallest woman being exactly d apart. We can do the same around the points t_2 and t_3 . As d increases, one of two things must happen. The first of these occurs in Figure 1; two or more of the subpopulations will impinge on each other. In Figure 1 this happens when the taller women

¹⁴Note that we cannot rely on the existence results of Kaneko and Wooders (1986). They assume a minimal degree of transferability; we assume none at all.

in the subpopulation around t_1 coincide with the shorter women in the subpopulation around t_2 . As soon as this occurs, the matchings formed so far, which by Lemma 3 together constitute a fixed matching of P^3 (the union of the three subpopulations around t_1 , t_2 , and t_3) relative to P^2 , form a part of a stable matching of P^2 and hence of P .

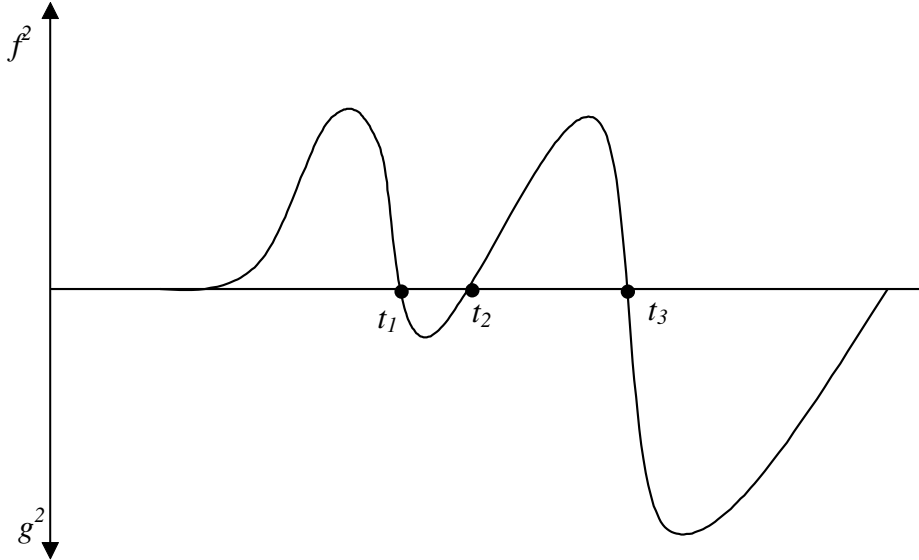


Figure 1: the density of types in the subpopulation $P^2 = P \setminus P^0$.

The type densities f^4 and g^4 of the remaining subpopulation $P^4 = P^2 \setminus P^3$ are illustrated in Figure 2, the essential point of which is that we now have only one point of intersection (as defined in Assumption 1). Technically, this point could lie within a range and Figure 2 takes it to be t_3 .

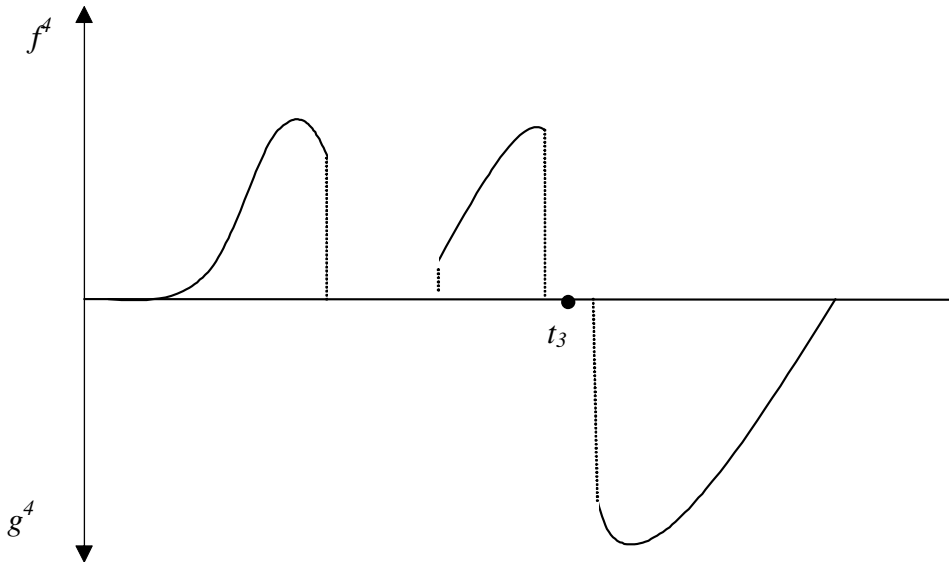


Figure 2: the densities of types in the subpopulation $P^4 = P^2 \setminus P^3$.

As all men in P^4 are shorter than t_3 and all women are taller than t_3 , we can easily construct a stable matching, which must display negative assortment.

The second possibility as d increases is that either the subpopulation containing the shortest agents or that containing the tallest agents cannot expand any further (as increasing d takes the

range of types beyond \underline{t} or \bar{t} . Then we are left with a subpopulation P^4 where the type densities intersect once fewer. In either case, this process of looking for fixed matchings can continue with a sequence $P^4, P^6, P^8 \dots$, with each element of the sequence having a pair of type densities that intersect fewer times than its predecessor's. Thus the sequence is finite and results in a stable matching, μ^* , of the whole population. We denote by θ^* the assignment induced by μ^*

That *any* stable matching of P can be derived from essentially a similar procedure implies the following:

Theorem 7 *P has a unique stable assignment, θ^* .*

The intuition is again straightforward. As the distance d between type pairs increases we draw in and match more and more of the population. Setting $d = 0$, we match an overlap of the population, possibly but not necessarily P^0 . For d positive but small, pairs close to each in type are matched, and the distribution of types in the remaining subpopulation is the same regardless of the actual identity of the agents matched so far. As d increases further, some agents may be obliged to match with partners of a very different type.

The pattern of matching that emerges from this process has a number of very distinctive characteristics. Firstly, there is a lower bound to the degree of positive sorting given by Ω , the measure of any overlap. The small subpopulations that are matched (e.g. those around t_1, t_2 , and t_3 in Figure 1) each display negative sorting, but there is a further degree of positive sorting as both the men and the women near t_1 are shorter than those near t_2 , who are in turn shorter than those near t_1 ; there is thus some positive sorting across subpopulations.

Secondly, an agent of a particular type may or may not be matched with a partner of the same type. Somewhat informally, if $f(s) < g(s)$, then almost all men of type s will match with a woman of type s , but only a proportion $f(s)/g(s)$ of women of type s will match with a man of type s . Almost all of the remainder $g(s) - f(s)$ will match with men of one particular type, say type t , but what t actually is cannot easily be calculated, and depends on the functions f and g . If s is very near an intersection s' of f and g then t will be just the other side of the intersection, so the mismatch will be small. But in Figure 2 the tallest woman is very badly matched, as she is paired with the shortest man. Overall, we can say that the locally scarce sex (men if $f(s) < g(s)$) get an ideal partner; some of the locally common sex get an ideal partner, but others don't, although they are not necessarily very badly matched.

4 MATCHING AND SORTING IN TWO SIMPLE CASES

Suppose that one type density is merely a rightward or leftward shift of the other. For example, $g(t) = f(t + x)$ for some $x > 0$; this would imply that the mean male height is greater than the mean female height but that the second and higher moments of the two distributions are the same. Suppose further that two densities are unimodal; then f and g intersect once, at some point t_1 in between the two modes. What does the pattern of matching look like?

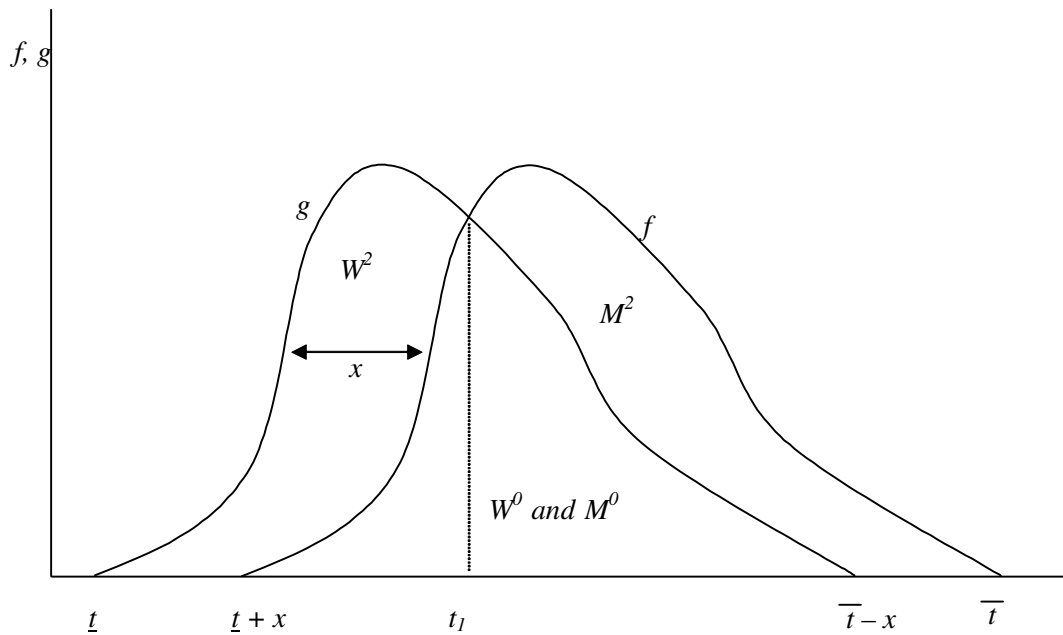


Figure 3: f is a rightward shift of g .

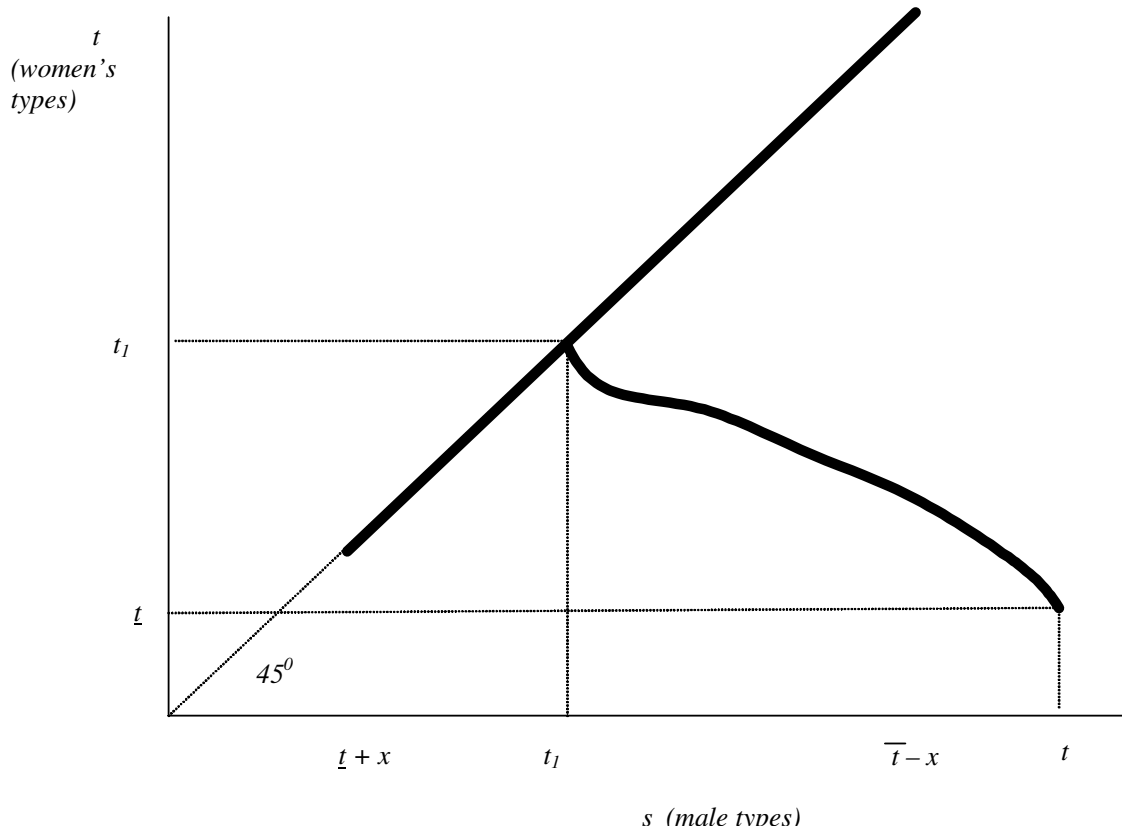


Figure 4: the support of the stable assignment when f is a rightward shift of g .

Figures 3 and 4 illustrate such a situation. There is an overlap between $\underline{t} + x$ and $\bar{t} - x$ so the subpopulation $P^0 = (M^0, W^0)$ - or one with the same distribution of types - can match like with like. The remaining subpopulation $P^2 = (M^2, W^2)$ - or one with the same distribution of types - consists of women shorter than t_1 and men taller than t_1 , who therefore match negatively. The resulting assignment is shown in Figure 4, where the heavy lines constitute the support of θ^* (i.e. the largest subset of $T \times T$ for which every open neighbourhood of every point in it has

positive measure). P^0 matches along the diagonal from $\underline{t} + x$ to $\bar{t} - x$ and P^2 along a negatively sloped line from (t_1, t_1) to (\bar{t}, \underline{t}) .

Suppose now that the two distributions are both unimodal and symmetric around a common mean x , but differ in their variance. For example if $g(t) = af(x + a(t - x))$, where $a > 1$, then there are more very short and very tall men, but more women closer to the average. Figure 5 illustrates such a situation.

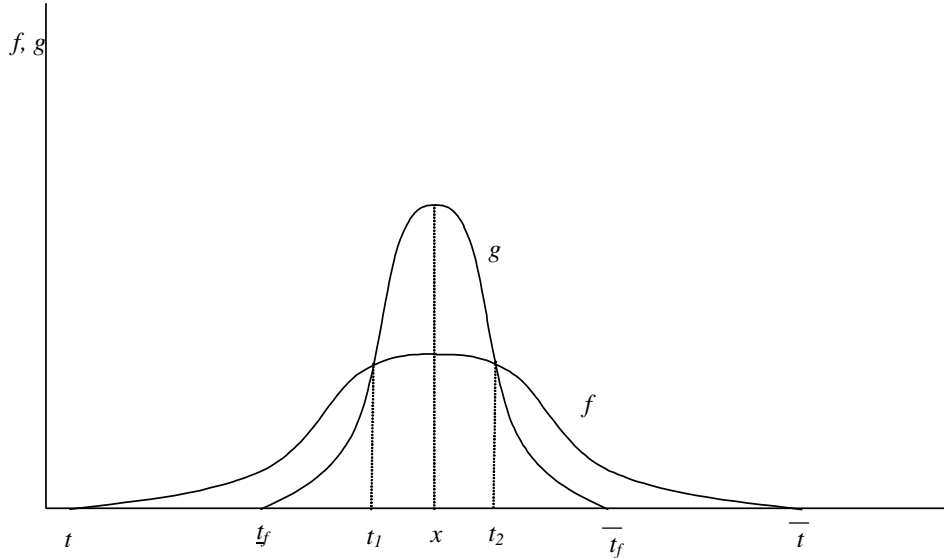


Figure 5: f has the same mean but a greater variance than g and they intersect twice.

There is an overlap between \underline{t}_f and \bar{t}^f , thus allowing like to match with like. When the overlap is taken away, we are left with a subpopulation of short men (\underline{t} to t_1), women of average height (t_1 to t_2) and tall men (t_2 to \bar{t}). The shorter women (t_1 to x) match with the shorter men, with negative sorting; the taller women (x to t_2) match with the taller men, with negative sorting. The support of the overall assignment is shown in Figure 6.

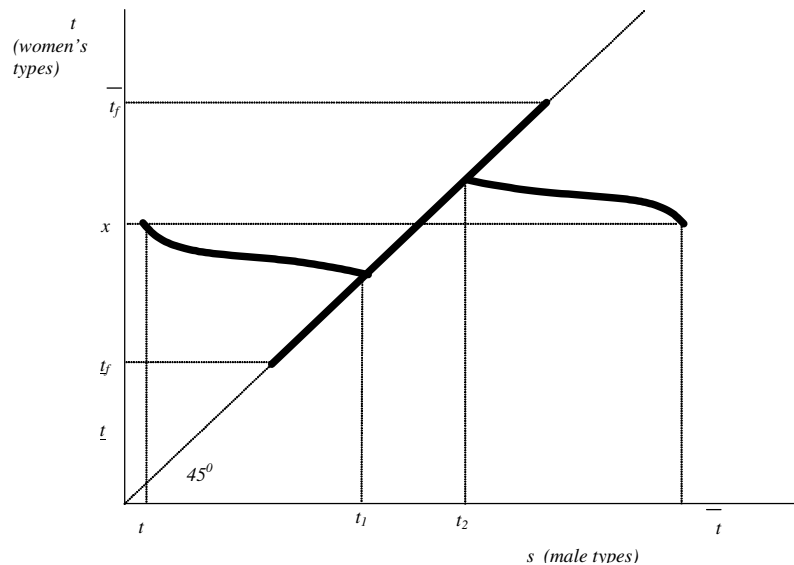


Figure 6: the support of the stable assignment when f has the same mean but a greater variance than g and they intersect twice.

5 Unbalanced populations and preferences for celibacy

We give here an informal account of two generalisations of the basic model as set out in Section 2. Suppose that $F(\bar{t}) > G(\bar{t})$, so there are more men than women. To derive a stable matching, we first match any overlapping subpopulation and then embark on the process of matching subpopulations with types either side of the intersections of $f - g$. Initially, we match pairs with similar types (i.e. when d is low, in the language of Section 3.3). As the process continues, the distance between the types of matched pairs increases, but as any match is better than none, we carry on until we run out of agents. If $F(\bar{t}) > G(\bar{t})$, this occurs when all women are matched. The mass $F(\bar{t}) - G(\bar{t})$ of unmatched men are, in an obvious sense, the least desirable; no woman would want to give up her partner for one of these men. Figure 7 illustrates; there is an overlap between \underline{t}_f and \bar{t}_g , leaving an unbalanced subpopulation of men with types from \underline{t}_g to t_1 and women with types from t_1 to \bar{t}_f . As there are more men only those from t_1 to x can be matched (negatively), leaving unmatched a set of tall men (from x to \bar{t}_f).

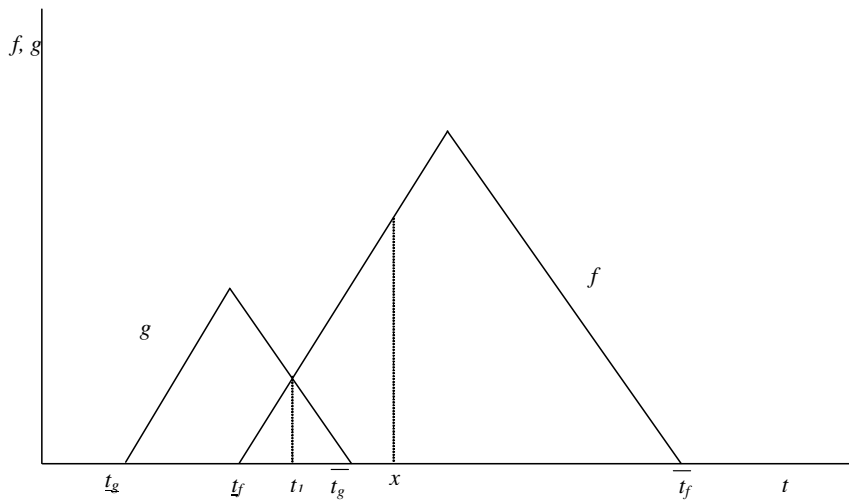


Figure 7: $\lambda_2(M) > \lambda_2(W)$. Men with types greater than x remain unmatched.

Preferences for celibacy are straightforward to introduce. Suppose all agents prefer no partner to one who differs in type from them by more than d' . Then the process of matching pairs who are increasingly badly matched will stop as soon as the distance d between types reaches d' . If agents differ in their 'reservation distance', the process is more complicated, depending on how reservation distance is distributed among types. In general, as d increases, some unmatched agents will drop out of the marriage market, preferring celibacy to a bad match. Figure 8 illustrates a case where f is uniform on $[\underline{t}_f, \bar{t}_f]$ and g is uniform on $[\underline{t}_g, \bar{t}_g]$ i.e. there is an overlap between \underline{t}_f and \bar{t}_g . All women and some men in this range are perfectly matched. Suppose that in the remaining subpopulation there is a subset of men, M^c , all with types greater than \bar{t}_g whose reservation distance is, for some reason, less than $\bar{t}_g - \underline{t}_f$. Then they each prefer celibacy to a woman of type less than \underline{t}_f , and once the overlap of P is matched they will drop out of the marriage market. Then not all women of type less than \underline{t}_f will find a partner. As shown in Figure 8, those men who are not quite so fussy as those in M^c will match, with negative sorting, with women with types from t' to \underline{t}_f , leaving unmatched a set of women W^c such that $\lambda_2(W^c) = \lambda_2(M^c)$.

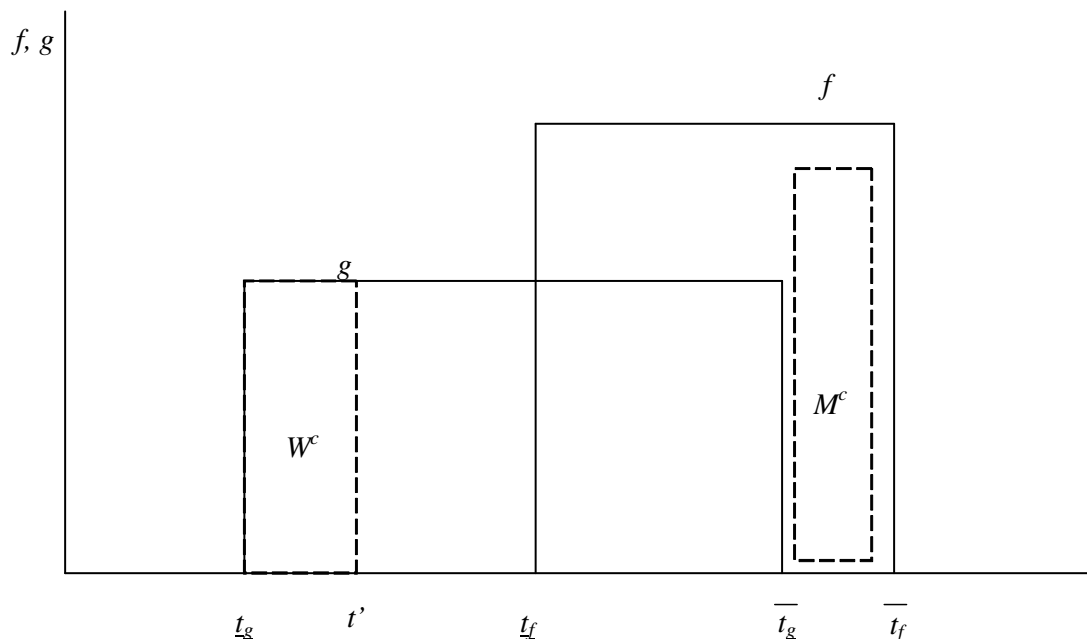


Figure 8: $\lambda_2(M) = \lambda_2(W)$ but the men in M^c prefer celibacy to matching with women with types less than \underline{t}_f . Agents in M^c and W^c remain unmatched.

6 CONCLUSION

This paper has shown that in the case of non-transferable utility, “like attracts like” does not necessarily imply that “like is matched with like”, nor does it imply any particular pattern of sorting. Complete positive assortment is an extremely special case that arises only when the distribution of types (or characteristics) is the same among men as it is among women. Complete negative assortment is also a special case, although immune to small perturbations in the two distributions; it arises in populations where all men are taller, or are of a greater type, than all women (or vice versa). In the general case, like is matched with like only to an extent given by the degree of overlap between the type distributions: if there are more men than women of type t then all type t women match with a type t man, but not all type t men match with a type t woman; thus there is no principle of “equal treatment”. This overlap puts a lower bound on the extent of positive assortment. The remainder of the population are not perfectly matched and consist of a number of subpopulations, within each of which there is negative assortment. However, there is some further positive assortment arising from the positive correlation across these subpopulations; for example if a set of very short men match (negatively) with a set of quite short women, and a set of quite tall women match (negatively) with a set of very tall men.¹⁵ The pattern of matching is thus potentially rich and complex. From an empirical point of view simple correlations are unlikely to pick this up.

Finally, a brief comment on the welfare properties of the equilibria analysed here. In the case of transferable utility, equilibria can be derived from the condition that the total output from matching is maximised. It is then but a brief step to show that matching will display positive assortment if the combined output of two matched agents is a supermodular function of their characteristics, whatever the distribution of types. In the case of non-transferable utility, stable

¹⁵In Clark (2003) I compute the correlation between men’s types and that of their partners for the case of uniform distributions, and derive an explicit relationship between the correlation coefficient (r) and the means and spreads of the two distributions.

matchings are Pareto efficient (almost by definition) but “combined output” is not a meaningful concept. Nevertheless we can use supermodularity to make some interesting welfare statements. Suppose that social welfare is the sum of individual utilities, and that an agent of type t when matched with a partner of type s has utility that is a concave function of $|s - t|$ e.g. $-(s - t)^2$. Then the social welfare arising from that match is also a concave function of $|s - t|$ and hence a supermodular function of t and s . Thus social welfare is maximised by positive assortment and minimised by negative assortment. But without transferable utility, stability is unconnected to social welfare. Unless the two type distributions are identical, social welfare will not be maximised in equilibrium.

APPENDIX

Proof of Theorem 1: By definition, $\hat{\mu}_1(s, x) = \alpha(s)$, and if $F = G$, then $\alpha(s) = s$. Thus $\hat{\mu}$ gives each agent their ideal partner; thus no pair not matched by $\hat{\mu}$ can block $\hat{\mu}$, so $\hat{\mu}$ is totally stable and hence stable.

Proof of Theorem 2 If $F = G$, then $\alpha(s) = s$, so the graph of α is the diagonal D . (a) Sufficiency. If $\theta_\mu = \theta^\alpha$ then $\theta_\mu(D) = 1$, implying that in all pairs matched by μ (except for a subpopulation P^0 of measure zero) the man and the woman are of the same type. If we take away from P those pairs in P^0 we form a subpopulation P' and a matching μ' , in accordance with Definition 2, such that in all pairs matched by μ' the man and the woman are of the same type; thus μ' is a totally stable matching of P' . Since $\lambda_2(P') = 1$, μ is stable.

(b) Necessity. Let $P_1 = (M_1, W_1)$ be the subpopulation of P such that M_1 and W_1 are the sets of men and women respectively paired by μ with a partner of his/her own type. Let F_1 and G_1 be the distribution functions for M_1 and W_1 ; then $F_1 = G_1$. Let $P_2 = (M_2, W_2)$, where $M_2 = M \setminus M_1$ and $W_2 = W \setminus W_1$, with distribution functions F_2 and G_2 ; then $F_2 = F - F_1 = G_2 = G - G_1$ and $\lambda_2(P_2) = 1 - \theta_\mu(D)$.

Consider any subpopulation $P' = (M', W')$, such that $\lambda_2(P') = 1$, and any matching μ' of P' such that $\mu'(m) = \mu(m)$ for $m \in M'$. Let $P'_2 = (M'_2, W'_2) = P' \cap P_2$ with distribution functions F'_2 and G'_2 . As $\lambda_2(P') = 1$, P' must include almost all of P_2 in the sense that $\lambda_2(P'_2) = \lambda_2(P_2)$; thus $F'_2 = G'_2 = F_2 = G_2$ and $\lambda_2(P'_2) = 1 - \theta_\mu(D)$.

Suppose now that $\theta_\mu \neq \theta^\alpha$; then $1 - \theta_\mu(D) > 0$, and P' contains a subpopulation, P'_2 , of positive measure, and hence with at least one man and one woman, such that no man or woman in P'_2 is matched by μ' with a partner of the his/her type. But because $F'_2 = G'_2$ and $\lambda_2(P'_2) > 0$, there must be at least one man m^* in M'_2 and one woman w^* in W'_2 who have the same type; then m^* and w^* can block μ' , which therefore is not totally stable. Hence it is not possible to find a subpopulation P' and a totally stable matching μ' that satisfy the requirements of Definition 2. Therefore μ is not stable if $\theta_\mu \neq \theta^\alpha$, so it is stable only if $\theta_\mu = \theta^\alpha$.

Proof of Theorem 3 Suppose that $\theta_\mu = \theta^\alpha$. If $F \neq G$, then α is not the identity function and its graph does not coincide with the diagonal i.e. $\Gamma_\alpha \neq D$. Since α is continuous, we can find a segment of Γ_α that lies either below or above D ; hence there exist intervals $[s_1, s_2] \in T_f$ and $[t_1, t_2] = [\alpha(s_1), \alpha(s_2)] \in T_g$, with $s_1 < s_2$, such that either $t_2 < s_1$ or $s_2 < t_1$. Let $M' = \{(s, x) \in M | s \in [s_1, s_2]\}$ and $W' = \{(t, y) \in W | t \in [t_1, t_2]\}$ (so that $W' = \mu(M')$) and suppose $t_2 < s_1$, so $t_1 < t_2 < s_1 < s_2$. Then apart from men with type s_2 , all men in M' prefer a woman with type t_2 to their own partners and apart from women of type t_1 all women in W' prefer a man with type s_1 to their own partners; so there exists a set of blocking pairs of measure $\lambda_2(M')$. A similar

result applies to any subsets $M'' \subset M', W'' \subset W'$ such that $W'' = \mu(M'')$ and $\lambda_2(M'') > 0$: apart from the tallest men and the shortest women, everyone can be part of a pair that blocks μ . This makes impossible any attempt, in accordance with Definition 2, to form from μ and P a matching μ' that is a totally stable matching of a population P' that has the same measure of men and women as P . Thus μ is not a stable matching of P . If $s_2 < t_1$ then a similar argument applies, with all but the tallest women and the shortest men able to be part of a blocking pair. Therefore if $\theta_\mu = \theta^\alpha$, μ is not stable, so μ is stable only if $\theta_\mu \neq \theta^\alpha$.

Proof of Theorem 4 (i) Let $m \in M$ and $w \in W$ be unmatched by $\tilde{\mu}$, where $m = (s, x)$ and $w = (t, y)$ and let $\tilde{\mu}_1(s, x) = t'$ and $\tilde{\mu}_1^{-1}(t, y) = s'$. Since no woman is taller than any man, $t \leq s$ and $t' \leq s'$. Suppose that (m, w) can block $\tilde{\mu}$; then (i) m prefers w to $\tilde{\mu}(m)$, which implies that $t' < t$; and (ii) w prefers m to $\tilde{\mu}^{-1}(w)$, which implies that $s < s'$. Thus $t' < t \leq s < s'$. Then the shorter man, m , is matched with the shorter woman, $\tilde{\mu}(s, x)$ and the taller man, $\tilde{\mu}^{-1}(t, y)$, with the taller woman, w , contrary to the negative assortment induced by $\tilde{\mu}$. Thus no pair can block $\tilde{\mu}$; therefore it is totally stable and hence stable.

(ii) (a) Sufficiency: Suppose $\theta_\mu = \theta^\beta$. Then θ_μ attaches positive measure only to subsets of T^2 that are subsets of Γ_β , the graph of β . Therefore the measure of the set of pairs matched by μ whose types lie off Γ_β must be zero; i.e. $\lambda_2(M^0) = 0$ where $M^0 = \{(s, x) \in M : (s, \mu_1(s, x)) \in T^2 \setminus \Gamma_\beta\}$. If we now form the population $P' = (M \setminus M^0, W \setminus \mu(M^0)) = (M', W')$, and consider the matching $\mu' : M' \rightarrow W'$ where $\mu'(m) = \mu(m)$ if $m \in M'$, then all pairs matched by μ' have types that lie on Γ_β .

Suppose that $m = (s, x) \in M'$ and $w = (t, y) \in W'$ can block μ' ; since no man is shorter than any woman, this implies that w is taller than m 's partner (i.e. $t > \mu'_1(m)$) and m is shorter than w 's partner (i.e. $\mu'_1^{-1}(w) > s$). This in turn implies that the shorter man, m , is matched by μ' with the shorter woman, $\mu'(m)$. Thus the pairs $(m, \mu'(m))$ and $(\mu'^{-1}(w), w)$ cannot both lie on Γ_β , a contradiction. Thus no pair $(m, w) \in M' \times W'$ can block μ' . This implies that μ' is a totally stable matching of P' and, since $\lambda_2(P') = 1$, that μ is a stable matching of P .

(ii) (b) Necessity: Suppose μ is stable; then there exists a population $P' = (M', W')$ and a totally stable matching μ' of P' such that $\theta_\mu = \theta_{\mu'}$. Since μ' is totally stable, it has no blocking pairs, so for any two men in M' , $m = (s, x)$ and $m' = (s', x')$ if $s < s'$ then $\mu'_1(s, x) \geq \mu'_1(s', x')$, since if $\mu'_1(s, x) < \mu'_1(s', x')$ then m and $\mu'(m')$ would prefer each other to $\mu'(m)$ and m' respectively (recall that all men are taller than all women) i.e. $(m, \mu'(m'))$ could block μ' . Thus a taller man (and by a similar argument a taller woman) does not have a taller partner; i.e. μ' induces weakly negative assortment (WNA) We now show that this implies SNA.

Suppose that $\mu'(s, x) = t$ for any $s \in [s_1, s_2]$. Then measure consistency of μ' requires that $\lambda_m([s_1, s_2]) \leq \lambda_w(t)$. But $\lambda_w(t) = 0$ and $\lambda_m([s_1, s_2]) = \int_{s_1}^{s_2} f(x)dx$. so $\lambda_m([s_1, s_2]) \leq \lambda_w(t)$ only if $s_1 = s_2$. This implies that only one type of man can be matched with a type t woman. Similarly, only one type of woman can be matched with a type s man. Thus the WNA established above implies that the taller agent has a strictly shorter partner. Denote by $\gamma(s)$ the type of woman matched by μ' to a type s man. Men taller than s are matched with women shorter than $\gamma(s)$, so $G(\gamma(s)) = (1 - F(s))$; thus $\gamma(s) = G^{-1}(1 - F(s))$ i.e. $\gamma = \beta$. Therefore $\theta_{\mu'} = \theta^\beta$. Thus if μ is stable $\theta_\mu = \theta^\beta$.

Proof of Lemma 2 Consider the subpopulation $P^0 = (M^0, W^0)$ where $M^0 = W^0 = M \cap W$. In both M^0 and W^0 the function $h = \min(f, g)$ describes the density of types, with distribution function $H : H(t) = \int_{[t, t]} h(x)dx$, and $\lambda_2(P_0) = \Omega$. Let $P_1 = (M_1, W_1)$ be the subpopulation of P such that M_1 and W_1 are the sets of men and women respectively paired by μ with a partner of his/her own type. Let F_1 and G_1 be the distribution functions for M_1 and

W_1 ; then by construction $F_1 = G_1$ and $\lambda_2(P_1) = \theta_\mu(D)$. Let $P_2 = (M_2, W_2)$, where $M_2 = M^0 \setminus M_1$ and $W_2 = W^0 \setminus W_1$, with distribution functions F_2 and G_2 ; then $F_2 = H - F_1$ and $G_2 = H - G_1$, so $F_2 = G_2$ and $\lambda_2(P_2) \geq \Omega - \theta_\mu(D)$.

Consider any subpopulation $P' = (M', W')$, such that $\lambda_2(P') = 1$, and any matching μ' of P' such that $\mu'(m) = \mu(m)$ for $m \in M'$. Let $P'_2 = (M'_2, W'_2) = P' \cap P_2$ with distribution functions F'_2 and G'_2 . As $\lambda_2(P') = 1$, P' must include almost all of P_2 in the sense that $\lambda_2(P'_2) = \lambda_2(P_2)$; thus $F'_2 = G'_2 = F_2 = G_2$, with $F'_2(\bar{t}) = G'_2(\bar{t}) \geq 1 - \theta_\mu(D)$.

Suppose that $\theta_\mu(D) \neq \Omega$; then $0 \leq \theta_\mu(D) < \Omega$, and P' contains a subpopulation P'_2 of positive measure, and hence with at least one man and one woman, such that no man or woman in P'_2 is matched by μ' with a partner of the his/her type. But because $F'_2 = G'_2$ and $\lambda_2(P'_2) > 0$, there must be at least one man m^* in M'_2 and one woman w^* in W'_2 who have the same type; then m^* and w^* can block μ' , which therefore is not totally stable. Hence it is not possible to find a subpopulation P' and a totally stable matching μ' that satisfy the requirements of Definition 2. Therefore μ is not stable if $\theta_\mu(D) \neq \Omega$, so if μ is stable if $\theta_\mu(D) = \Omega$.

Proof of Theorem 5 Part A (Necessity). Suppose μ is a stable matching of P . Then, from Lemma 2, $\theta_\mu(D) = \Omega$. This implies the existence of sets $M' \subseteq M$ and $W' \subseteq W$, where $\mu(M') = W'$, such that (i) $\lambda_2(M') = \lambda_2(W') = \Omega$; and (ii) $\mu_1(s, x) = s$ for almost all $(s, x) \in M'$ i.e. $\lambda_2(\{(s, x) \in M' : (s, \mu_1(s, x)) \in D\}) = \Omega$. Thus (M', W') must be an overlapping subpopulation of P , and, by Lemma 1, μ' , defined by $\mu'(m) = \mu(m)$ for $m \in M'$, is a stable matching of P' . This shows the necessity of (a). If μ maps M onto W , and M' onto W' , then it must map $M'' = M \setminus M'$ onto $W'' = W \setminus W'$. Define μ'' so that $\mu''(m) = \mu(m)$ for $m \in M''$. If μ is a stable matching of P then P has a subpopulation $\hat{P} = (\hat{M}, \hat{W})$ such that (i) $\lambda_2(\hat{M}) = 1$ and (ii) $\hat{\mu}$ is a totally stable matching of \hat{P} where $\hat{\mu}(m) = \mu(m)$ for $m \in \hat{M}$. Let $\hat{M}'' = \hat{M} \cap M''$ and $\hat{W}'' = \hat{W} \cap W''$. As $\hat{\mu}$ is totally stable, no pair in $\hat{M} \times \hat{W}$ can block $\hat{\mu}$; in particular, as $\hat{M}'' \subseteq \hat{M}$ and $\hat{W}'' \subseteq \hat{W}$, no pair in $\hat{M}'' \times \hat{W}''$ can block $\hat{\mu}$; thus no pair in $\hat{M}'' \times \hat{W}''$ can block $\hat{\mu}''$, where $\hat{\mu}''(m) = \hat{\mu}(m)$ for $m \in \hat{M}''$ (since a blocking of $\hat{\mu}''$ would also be a blocking of $\hat{\mu}$). Thus $\hat{\mu}''$ is a totally stable matching of $\hat{P}'' = (\hat{M}'', \hat{W}'')$. But $\lambda_2(\hat{M}'') = \lambda_2(M'') = \lambda_2(\hat{W}'') = \lambda_2(W'')$; and $\hat{\mu}''(m) = \mu''(m)$ for $m \in \hat{M}''$; hence μ'' is a stable matching of $P'' = (M'', W'')$. This shows the necessity of (b).

Part B (Sufficiency). Suppose that for some matching μ and for some overlapping subpopulation $P' = (M', W')$ of P , conditions (a) and (b) are satisfied by matchings μ' and μ'' respectively. Since μ' is a stable matching of the overlap P' , then Lemma 2, $\lambda_2(\{(s, x) \in M' : (s, \mu'_1(s, x)) \in D\}) = \Omega$, so matched pairs except for a set of measure zero have types that lie on D . Thus it is possible to form $\hat{P}' = (\hat{M}', \hat{W}')$, a subpopulation of P' , such that (i) $\lambda_2(\hat{M}') = \lambda_2(M')$ and (ii) for all pairs such that $(m, \mu'(m)) \in \hat{M}' \times \hat{W}'$, m has the same type as $\mu(m)$. Since μ'' is stable, P'' has a subpopulation $\hat{P}'' = (\hat{M}'', \hat{W}'')$ such that $\hat{\mu}''$ is a totally stable matching of \hat{P}'' , where (i) $\lambda_2(\hat{M}'') = \lambda_2(M'')$ and (ii) $\hat{\mu}''(m) = \mu''(m)$ for $m \in \hat{M}''$. Now consider $\hat{P} = (\hat{M}, \hat{W})$, where $\hat{M} = \hat{M}' \cup \hat{M}''$ and $\hat{W} = \hat{W}' \cup \hat{W}''$; clearly \hat{P} is a subpopulation of P and since $\hat{M}' \cap \hat{M}'' = \emptyset$ (the empty set), $\lambda_2(\hat{M}') = \lambda_2(M')$, $\lambda_2(\hat{M}'') = \lambda_2(M'')$, and $\lambda_2(M') + \lambda_2(M'') = 1$, then $\lambda_2(\hat{M}) = 1$; similarly $\lambda_2(\hat{W}) = 1$. Now, consider the matching $\hat{\mu}$ where $\hat{\mu}(m) = \mu(m)$ for $m \in \hat{M}$. To show that μ is a stable matching of P , we need only show that $\hat{\mu}$ is a totally stable matching of \hat{P} . Suppose it is not; then there exists a pair $(\bar{m}, \bar{w}) \in \hat{M} \times \hat{W}$ that can block $\hat{\mu}$; since $\hat{\mu}$ matches every man in \hat{M}' with a woman of the same type, then $\bar{m} \notin \hat{M}'$ and $\bar{w} \notin \hat{W}'$; thus $(\bar{m}, \bar{w}) \in \hat{M}'' \times \hat{W}''$. But $\hat{\mu}(m) = \hat{\mu}''(m)$ for $m \in \hat{M}''$, so if $(\bar{m}, \bar{w}) \in \hat{M}'' \times \hat{W}''$ this would imply that (\bar{m}, \bar{w}) can block $\hat{\mu}''$; but $\hat{\mu}''$ is a totally stable matching of $\hat{P}'' = (\hat{M}'', \hat{W}'')$, a contradiction. This shows the sufficiency of (a) and (b).

Proof of Lemma 3 As $\mu^\#$ and μ^\sim are both stable, any pair that can block μ^* must have one partner in $P^\#$ and one in P^\sim ; but by the fixity of $\mu^\#$ any m in $M^\#$ prefers $\mu^\#(m)$ to any

w in W^\sim and any w in $W^\#$ prefers $\mu^{\#-1}(w)$ to any m in M^\sim . Thus μ^* cannot be blocked and is therefore stable. Furthermore, by the fixity of $\mu^\#$ and μ^\sim any m in M^* prefers $\mu^*(m)$ to any w in $\widehat{W} \setminus W^*$ and any w in W^* prefers $\mu^{*-1}(w)$ to any m in $\widehat{M} \setminus M^*$. Hence μ^* is a fixed matching of P^* relative to \widehat{P} .

Proof of Lemma 4 If $\mu^\#$ is a fixed matching of $P^\#$ then it is stable. If μ^\sim is also stable then we can form totally stable matchings $\mu^{\#'} and $\mu^{\sim'}$ of $P^{\#'} and $P^{\sim'}$ respectively, where $P^{\#'} = (M^{\#'}, W^{\#'})$, $P^{\sim'} = (M^{\sim'}, W^{\sim'})$, $M^{\#'} \subseteq M^\#$, $W^{\#'} \subseteq W$, $M^{\sim'} \subseteq M^\sim$, $W^{\sim'} \subseteq W$, $\lambda_2(P^{\sim'}) = \lambda_2(P^\sim)$ and $\lambda_2(P^{\#'}) = \lambda_2(P^\#)$. Now form $P' = (M', W') = (M^{\#'} \cup M^{\sim'}, W^{\#'} \cup W^{\sim'})$; thus $\lambda_2(P') = \lambda_2(\widehat{P})$; form the matching $\mu' : M' \rightarrow W'$: where $\mu'(m) = \mu^{\#'}(m)$ if $m \in M^{\#'}$ and $\mu'(m) = \mu^{\sim'}$ if $m \in M^{\sim'}$. Any pair $(m, w) \in M' \times W'$ that blocks μ' is an element either of $M^{\#'} \times W^{\#'}$, or of $M^{\sim'} \times W^{\sim'}$, or of $M^{\#'} \times W^{\sim'}$, or of $M^{\sim'} \times W^{\#'}$. The first and second possibilities contradict the total stability of $\mu^{\#'}$ and $\mu^{\sim'}$ respectively, the third implies that $m \in M^\#$ prefers a woman, w , not in $W^\#$ to a $\mu^{\#'}(m)$, a contradiction since $\mu^\#$ is a fixed matching of $P^\#$ and the fourth possibility implies that w prefers a man, m , not in $M^\#$ to a man, $\mu^{\#'-1}(w)$ in $M^\#$, a similar contradiction. Thus no pair in $M' \times W'$ can block μ' , which is therefore totally stable. But P' is a subpopulation of \widehat{P} with $\lambda_2(P') = \lambda_2(\widehat{P})$, and $\mu'(m) = \mu(m)$ if $m \in M'$. Therefore μ' fulfils the requirements of Definition 1 and μ is stable.$$

Proof of Theorem 6 If $F = G$, then by Theorem 1 $\widehat{\mu}$ is a stable matching of P . From now on, suppose $F \neq G$, so $f \neq g$. We construct a stable matching μ^* . For $m \in M$, $\mu^*(m) = m$; this uses ι^0 to match the overlapping subpopulation P^0 . We are now left with a subpopulation $P^2 = (M^2, W^2)$ with no overlap, the densities of male and female types being given by $f^2 = f - h$ and $g^2 = g - h$ respectively, with distribution functions F^2 and G^2 . From Theorem 5, it is sufficient to find a stable matching of P^2 . Since $f \neq g$, we now use Assumption 1. Without loss of generality suppose that in part (a) of Assumption 1 we have (i) for all $t \in T_i$, $k(t) \geq 0$ if i is odd and $k(t) \leq 0$ if i is even. Recall that $h = \min(f, g)$ and $k = f - g$; then from part (b) of Assumption 1, if i is odd $f^2(t) \geq 0$ and $g^2(t) = 0$ for all $t \in T_i$, with $\int_{T_i} f^2(x) dx > 0$ and if i is even $f^2(t) = 0$ and $g^2(t) \geq 0$ for all $t \in T_i$, with $\int_{T_i} g^2(x) dx > 0$.

Consider the point t_i , the intersection of T_i and T_{i+1} , and suppose that i is odd. We construct a subpopulation $P_i^\dagger = (M_i^\dagger, W_i^\dagger)$ of men with types in T_i and women with types in T_{i+1} as follows: since T_{i+1} is of positive length, we can choose a number $t_i^+ \in T_{i+1}$ such that there exists $t_i^0 \in T_i$ and

$$\int_{t_i^0}^{t_i} f^2(x) dx = \int_{t_i}^{t_i^+} g^2(x) dx.$$

This equation may be satisfied by more than one value of t_i^0 , so let $t_i^- = \max\{t_i^0 \in T_{i+1}, \int_{t_i^-}^{t_i} f^2(x) dx = \int_{t_i}^{t_i^+} g^2(x) dx\}$. Now let $M_i^\dagger = \{(s, x) \in M^2 | t_i^- \leq s \leq t_i\}$ and $W_i^\dagger = \{(t, y) \in W^2 | t_i \leq s \leq t_i^+\}$. t_i^- is a continuous non-increasing function of t_i^+ , so $\delta = t_i^+ - t_i^-$ is continuous and strictly increasing in t_i^+ . As $\lambda_2(P_i^\dagger) = \int_{t_i^-}^{t_i^+} g^2(x) dx$, $\lambda_2(P_i^\dagger)$ is continuous and non-decreasing in t_i^+ , so we may take $\lambda_2(P_i^\dagger)$ to be a continuous non-decreasing function, γ_i , of δ . If $\gamma_i(0) = 0$; as δ increases, $\gamma_i(\delta)$ weakly increases and the sets M_i^\dagger and W_i^\dagger expand (or do not shrink) in measure. Eventually either $t_i^+ = t_{i+1}$, or $t_i^- = t_{i-1}$, this sets an upper limit to δ , denoted δ_i . By part (b) of Assumption 1, $\gamma_i(\delta_i) > 0$.

If i is even, we follow a similar procedure, except that women have types in T_i and men types

in T_{i+1} , so that

$$\int_{t_i^0}^{t_i} g^2(x)dx = \int_{t_i}^{t_i^+} f^2(x)dx.$$

We can still write $\lambda_2(P_i^\dagger) = \gamma_i(\delta)$, where γ_i is a non-decreasing function.

We now construct a stable matching of P_i^\dagger . We start by constructing a totally stable matching of a subpopulation $P_i^{\dagger'} = (M_i^{\dagger'}, W_i^{\dagger'})$ in accordance with Definition 1. Note from the definition of P^2 that if $f^2(t) = 0$ then $f(t) \leq g(t)$ and M_i^\dagger contains no agents of type t_i (they are all in M^0) and if $g^2(t) = 0$ then $g(t) \leq f(t)$ and W_i^\dagger contains no agents of type t_i (they are all in W^0). Let F_i^\dagger and G_i^\dagger be the type distributions for M_i^\dagger and W_i^\dagger respectively. We look for pairs (s, t) such that $G_i^\dagger(t) = 1 - F_i^\dagger(s)$. If $G_i^\dagger(t) = 1 - F_i^\dagger(s)$ for $t \in [t_1, t_2]$ (where $t_1 < t_2$) then $g^2(t) = 0$ for $t \in [t_1, t_2]$ and W_i^\dagger contains no agent of type $t \in [t_1, t_2]$. We exclude from $M_i^{\dagger'}$ men with type s such that $G_i^\dagger(t) = 1 - F_i^\dagger(s)$ has multiple solutions. Then $G_i^\dagger(t) = 1 - F_i^\dagger(s)$ has a unique solution for any $(s, x) \in M_i^{\dagger'}$, so $\lambda_2(M_i^{\dagger'}) = \lambda_2(W_i^\dagger)$ (as W_i^\dagger contains no agent of type t which are part of a multiple solution to $G_i^\dagger(t) = 1 - F_i^\dagger(s)$). Thus $\lambda_2(M_i^{\dagger'}) = \lambda_2(M_i^\dagger)$. By a similar argument we exclude from $W_i^{\dagger'}$ women with type t such that $G_i^\dagger(t) = 1 - F_i^\dagger(s)$ has multiple solutions in s ; thus $\lambda_2(W_i^{\dagger'}) = \lambda_2(W_i^\dagger)$. Hence $\lambda_2(P_i^{\dagger'}) = \lambda_2(P_i^\dagger)$.

If $G_i^\dagger(t) = 1 - F_i^\dagger(s)$ for some $(s, x) \in M_i^{\dagger'}$ and some $(t, y) \in W_i^{\dagger'}$, we write $t = \beta_i^\dagger(s)$. Then $\mu_i^{\dagger'}$ is a matching of $P_i^{\dagger'}$, where $\mu_i^{\dagger'}(s, x) = (\beta_i^\dagger(s), f(\beta_i^\dagger(s)) + \frac{[x-g(s)]g^2(\beta_i^\dagger(s))}{f^2(s)})$. Total stability follows immediately from the fact that β_i^\dagger is a strictly decreasing function and the men in M_i^\dagger are either all taller or all shorter than any woman in W_i^\dagger . To complete the stable matching of P_i^\dagger we need to match those agents not in $P_i^{\dagger'}$. It is possible that although both $M_i^\dagger \setminus M_i^{\dagger'}$ and $W_i^\dagger \setminus W_i^{\dagger'}$ are of measure zero one set has no members, so we cannot just match men in $M_i^\dagger \setminus M_i^{\dagger'}$ with women in $W_i^\dagger \setminus W_i^{\dagger'}$. Instead we amend the matching $\mu_i^{\dagger'}$ (to μ_i^\dagger) so that a type s man in $M_i^{\dagger'}$ is still matched with a type $\beta_i^\dagger(s)$ woman in $W_i^{\dagger'}$ but we 'make room' for members of $P_i^\dagger \setminus P_i^{\dagger'}$. Specifically, since $\lambda_2(M_i^\dagger) > 0$ and $\lambda_2(M_i^\dagger \setminus M_i^{\dagger'}) = 0$ then for each s' such that there exists some $(s', x') \in M_i^\dagger \setminus M_i^{\dagger'}$ we can find $s = \phi(s')$ such that there exists some $(s, x) \in M_i^\dagger$, and such that if $s' \neq s''$ then $\phi(s') \neq \phi(s'')$. Similarly for each t' such that there exists some $(t', y') \in W_i^\dagger \setminus W_i^{\dagger'}$ we can find $t = \psi(t')$ such that there exists some $(t, y) \in W_i^\dagger$, and such that if $t' \neq t''$ then $\psi(t') \neq \psi(t'')$. μ_i^\dagger is constructed as follows: If $s = \phi(s')$ for some $(s', x') \in M_i^\dagger \setminus M_i^{\dagger'}$ then $\mu_i^\dagger(s, x) = (\beta_i^\dagger(s), f(\beta_i^\dagger(s)) + \frac{[x-g(s)]g^2(\beta_i^\dagger(s))}{f^2(s)+f^2(s')})$ for $(s, x) \in M_i^\dagger$ and $\mu_i^\dagger(s, x) = (\beta_i^\dagger(s), f(\beta_i^\dagger(s)) + \frac{[x-g(s')+f^2(s)]g^2(\beta_i^\dagger(s))}{f^2(s)+f^2(s')})$ for $(s, x) \in M_i^\dagger \setminus M_i^{\dagger'}$. If $t = \psi(t')$ for some $(t', y') \in W_i^\dagger \setminus W_i^{\dagger'}$ then $\mu_i^{\dagger-1}(t, y) = (\beta_i^{\dagger-1}(t), g(\beta_i^{\dagger-1}(t)) + \frac{[y-f(s)]g^2(\beta_i^{\dagger-1}(t))}{g^2(t)+g^2(t')})$ for $(t, y) \in W_i^\dagger$ and $\mu_i^{\dagger-1}(t, y) = (\beta_i^{\dagger-1}(t), g(\beta_i^{\dagger-1}(t)) + \frac{[y-f(s')+g^2(t)]f^2(\beta_i^{\dagger-1}(s))}{g^2(t)+g^2(t')})$ for $(t, y) \in W_i^\dagger \setminus W_i^{\dagger'}$. Otherwise $\mu_i^\dagger(s, x) = \mu_i^{\dagger'}(s, x)$. Then μ_i^\dagger is stable, as that part of it which matches $P_i^{\dagger'}$ is totally stable.

Starting from $\delta = 0$, as δ increases the subpopulations P_i^\dagger increase in measure from zero, and eventually one of two possibilities must occur: either (i) they start to impinge on each other, in the sense that for some i , then M_i^\dagger and M_{i+1}^\dagger (resp. W_i^\dagger and W_{i+1}^\dagger) have elements in common if i is odd (resp. even); or (ii) $t_1^- = t_0$ or $t_n^+ = t_n$. This may happen even if $\delta < \min_{1 \leq i \leq n-1} \delta_i$. We define $\delta(P_2)$ as the point at which this first occurs i.e. if $\delta = \delta(P_2)$ then $t_0 \leq t_1^-$, $t_{n-1}^+ \leq t_n$ and $t_i^+ \leq t_{i+1}^-$ for $i = 1, \dots, n-2$, with at least one of these n weak inequalities holding as an equality.

Consider now $P_i^\dagger, \gamma_i(\delta)$, and μ_i^\dagger for $i = 1, \dots, n-1$. and suppose that $\delta < \delta(P_2)$. Because of

the alternating nature of the subpopulations P_I^\dagger (if the men in P_I^\dagger are taller than the women then the opposite is true for P_{I+1}^\dagger and P_{I-1}^\dagger) then for all $(s, x) \in M_i^\dagger$, $|s - t| > \delta$ for any $(t, y) \notin W_i^\dagger$. But by construction $|s - t| \leq \delta$ for any $(s, x) \in M_i^\dagger$ and any $(t, y) \in W_i^\dagger$, including $\mu_i^\dagger(s, x)$. Thus μ_i^\dagger is a fixed matching of P_i^\dagger relative to P_2 . Similarly μ_{i+1}^\dagger is a fixed matching of P_{i+1}^\dagger relative to P_2 . Let μ^\dagger be the matching of $P^\dagger = (\cup_{i=1}^n M_i^\dagger, \cup_{i=1}^n W_i^\dagger)$, where $\mu^\dagger(m) = \mu_i^\dagger(m)$ if $m \in M_i^\dagger$. Then by repeated application of Lemma 3, μ^\dagger is a fixed matching of P^\dagger relative to P and by Lemma 4 to find a stable matching of P_2 it remains to find a stable matching of the subpopulation $P^4 = (M_2 \setminus M^\dagger, W_2 \setminus W^\dagger)$.

Suppose now $\delta = \delta(P_2)$. We consider two possibilities, at least one of which must occur: (i) $t_i^+ = t_{i+1}^-$ for at least one value of $i \leq n - 2$. Then the two subpopulations P_i^\dagger and P_{i+1}^\dagger just touch. To ensure agents of type t_i^+ are in only one of these subpopulations, we assign them to P_i^\dagger if i is odd (and correspondingly exclude agents of type t_i^- from P_i^\dagger) and to P_{i+1}^\dagger if i is even (and exclude agents of type t_{i+1}^+ from P_{i+1}^\dagger); this has no effect on the measure of P_i^\dagger and P_{i+1}^\dagger , and leaves μ_i^\dagger and μ_{i+1}^\dagger as fixed matchings and hence μ^\dagger as a fixed matching of P^\dagger relative to P_2 ; (ii) $t_0 = t_1^-$ or $t_{n-1}^+ = t_n$ or both

In case (i), P_4 is formed from P_2 by taking out a subpopulation P^\dagger which includes at least two adjacent (in type space) subpopulations P_i^\dagger and P_{i+1}^\dagger such that $t_i^+ = t_{i+1}^-$. We are therefore left with a subpopulation with density functions f^4 and g^4 that now cross no more than $n - 3$ times, rather than $n - 1$. In case (ii), we are left with density functions that cross no more than $n - 2$ times. More formally, let $k^4 = f^4 - g^4$. Then there exists a finite number $n' < n$ of intervals $S_1 = [s_0, s_1], S_2 = [s_1, s_2], S_3 = [s_2, s_3], \dots, S_{n'} = [s_{n'-1}, s_{n'}]$, where $s_0 = \underline{t}$ and $s_{n'} = \bar{t}$, all of positive length, such that (a) either (i) for all $s \in S_i$, $k(s) \geq 0$ if i is odd and $k(s) \leq 0$ if i is even; or (ii) for all $s \in S_i$, $k(s) \leq 0$ if i is odd and $k(s) \geq 0$ if i is even; and (b) for $i = 1, \dots, n'$, $\int_{S_i} k(x) dx \neq 0$ if $n' \neq 0$. There are two differences between this result and Assumption 1. First, we allow for $n' = 0$. In that case we have matched all of P and the Theorem is proved. Secondly, the types of any couple in P^4 are more than $\delta(P_2)$ apart, so the end-points of the intervals S_i are not uniquely defined.

Just as we formed a fixed matching μ^\dagger of $P^\dagger \subseteq P^2$, and then formed $P^4 = P^2 \setminus P^\dagger$, we can now form a fixed matching of $P^5 \subseteq P^4$. The analysis is the same as above. The end result is a subpopulation $P^6 = P^4 \setminus P^5$ (possibly with no agents, in which case the Theorem is proved) with type density functions that cross (in the sense of Assumption 1) no more than $n' - 2$ times.

Clearly, this process can continue until the population is fully matched; at each stage we add a fixed matching of a subpopulation, the total of which comprise a stable matching μ^* of P .

Proof of Theorem 7 Let μ be a stable matching of P and suppose that $\theta_\mu \neq \theta^*$. From Lemma 2 we know that $\theta_\mu(D) = \theta^*(D) = \Omega$, so θ_μ and θ^* differ only away from the diagonal D . We now consider a maximal ‘‘corridor’’ around D . Let $C(d) = \{(s, t) \in T \times T : |s - t| \leq d\}$, and let $d^* = \max\{d : \theta_\mu(Q) = \theta^*(Q) \text{ for all } Q \subseteq C(d)\}$. Then within $C(d^*)$ θ_μ and θ^* agree, Points within $C(d^*)$ represent matched pairs whose types are no more than d^* apart; outside $C(d^*)$ they are more than d^* apart.

Let $P_\Delta^* = (M_\Delta^*, W_\Delta^*)$ (resp. $P_\Delta = (M_\Delta, W_\Delta)$) be the subpopulation of P consisting of agents whose types are more than d^* from those of their partners, when matched by μ^* (resp. μ). As θ_μ and θ^* agree within $C(d^*)$, the distribution of male and female types is the same in P_Δ^* as in P_Δ . Let μ_Δ^* and μ_Δ be the matchings of P_Δ^* and P_Δ respectively such that $\mu_\Delta^*(m) = \mu^*(m)$ if $m \in M_\Delta^*$ and $\mu_\Delta(m) = \mu(m)$ if $m \in M_\Delta$. Since μ^* and μ are both stable, so are μ_Δ^* and μ_Δ .

For any $d > d^*$ $\theta_\mu(Q) \neq \theta^*(Q)$ for some $Q \subseteq (C(d) \setminus C(d^*))$ and hence $\theta_{\mu_\Delta}(Q) \neq \theta_{\mu_\Delta^*}(Q)$ for some $Q \subseteq (C(d))$. Without loss of generality²¹ suppose that such a Q lies above the diagonal

(where $t > s$). Then we can find some t^* such that $\theta_{\mu_\Delta}(Q) \neq \theta_{\mu_\Delta^*}(Q) > 0$ for some $Q \subseteq X$, where $X = \{(s, t) \in T \times T : t > t^* > s \text{ and } d > t - s > d^*\}$ (note that as t^* varies from \underline{t} to \bar{t} X covers all of $C(d) \setminus C(d^*)$). Within X , μ^* and μ_Δ^* must display negative sorting (as all women are taller than all men); more precisely, the matching within X displayed by μ^* and μ_Δ^* is the outcome of some part of the process by which μ^* is constructed the proof of Theorem 6. μ_Δ^* therefore matches almost all those women in P_Δ^* with types in a range denoted T_X with almost all those men in P_Δ^* with types in a range denoted S_X , where $\lambda_w(T_X) = \lambda_m(S_X) = \theta^*(X)$.

How does μ_Δ match this same mass of men in with types in S_X ? It cannot be via negative sorting with the $\theta^*(X)$ women in P_Δ with types in T_X , otherwise we would have $\theta_{\mu_\Delta}(Q) = \theta_{\mu_\Delta^*}(Q) > 0$ for any $Q \subseteq X$. Then there must be a subset M' , of positive measure, of men in P_Δ , with types in S_X , who are matched with women with types outside T_X and for measure consistency there must be a subset W' , of positive measure, of women in P_Δ , with types in T_X , who are matched with men with types outside S_X (as otherwise we would again have negative sorting in equilibrium and therefore $\theta_\mu(Q) = \theta^*(Q)$ for any $Q \subseteq X$). The men in M' cannot match with women whose types are within d^* of their own as this would contradict the stability of μ_Δ^* : by definition P_Δ^* has excluded all those who are able to match with a partner whose type is within d^* of their own. Thus the men in M' match with women whose types are more than d^* away from their own. If they match with women with types greater than those in T_X then they are worse off (as now their partners are even taller than them); if they match with women whose types are less than those in T_X then these female types must almost all be less than $\min(s \in S_X) - d^*$, and thus a positive measure of men must be worse off. By a similar argument the women in W' match with men whose types are more than d^* away from their own, and a positive measure of women must be worse off. The women and men who are worse off can match according to the measure given by μ^* , be better off than under μ_Δ^* , and therefore block μ_Δ^* . Hence μ_Δ^* cannot be stable. Thus $\theta_\mu = \theta^*$.

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