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# Noisy Talk*

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**Abstract**

We examine the possibilities for communication between agents with divergent preferences in a noisy environment. Taking Crawford and Sobel's [4] (noiseless) communication game as a reference point, we study a model in which there is a probability $\epsilon \in (0,1)$ that the received message is a random draw from the entire message space, independent of the actual message sent by the sender. Just as in the CS model, we find that all equilibria are interval partitional; but unlike in CS, coding (the proportion of the message space used by any given set of types) is of critical importance. Via the appropriate coding scheme, one can construct equilibria that induce finitely many, a countable infinity or even an uncountable infinity of actions. Furthermore, for a given number of actions, there is typically a continuum of equilibria that induce that many actions. Surprisingly, the possibility of error can improve the prospects for communication. We show that for small noise levels there is a simple class of equilibria that are almost always welfare superior to the best CS equilibrium. There exists an optimal noise level for which these equilibria achieve the efficiency bound for general communication devices. Furthermore, for a range of biases introducing any amount of noise can be beneficial.

Keywords. Communication, information transmission, cheap talk, noise.
JEL Classification. C72, D82, D83.

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# 1 Introduction

In many situations of economic interest, there is a division of labor between those who wield the power to make decisions and those who are best qualified to anticipate the consequences, the experts. Consequently, the decision makers often take advice from the experts before acting. Examples include legislators holding committee hearings and consulting with lobbyists to learn more about the likely impact of various policies under consideration; CEOs hiring management consultants before making major corporate decisions; and investors seeking the help of stockbrokers and financial advisors when considering different financial portfolios.

All of these situations share a common theme. The expert being consulted has an interest in the outcome of the decision, and yet typically has preferences that do not coincide with those of the decision maker — she is biased. As a result, the expert may have incentives to exaggerate in order to mislead the decision maker into making a choice more favorable to her. Conversely, the decision maker must interpret any advice given in the light of this bias.

The seminal analysis of strategic information transmission between a biased expert and an uninformed decision maker was provided by Crawford and Sobel [4]. In their model, the expert, or *sender*, learns the value of some payoff-relevant *state of nature*, before sending a costless (*cheap talk*) message to the decision maker, or *receiver*. The receiver then takes an action that affects the payoffs of both parties. The optimal action depends on the state of nature, but the sender and receiver disagree about what this optimal action is. A key result of the paper is that this divergence of interest has an adverse effect on the flow of information. More precisely, they show that, under fairly general assumptions about preferences, every equilibrium of the game is *interval partitional*: the sender's message reveals only in which of a finite number of intervals of the state space the true state lies. Furthermore, the number of elements distinguished has some upper bound that depends on the degree of bias of the expert. If the bias is too large, no meaningful communication is possible. With additional restrictions on preferences, Crawford and Sobel show that this bound is decreasing in the size of the bias, i.e. more information can be transmitted when preferences are more closely aligned.

Traditional information theory, pioneered by Shannon [22], abstracts away from strategic considerations raised by sender/receiver conflict of interest, and focuses on the process of information transmission itself. For Shannon, "The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point". His goal was to provide mechanisms that would transmit messages as faithfully as possible, in the presence of some fundamental source of noise. In Shannon's original model, the only source of noise is in the *channel* through which signals (translated messages) are passed from sender to receiver[1], but in a broader context we may think of errors as arising in the translation process as well. Suppose, for example, that Alice is trying to pass on some information to Bob by means of verbal communication. Three sources of error are possible: she may fail to choose appropriate words to express her thoughts; he may not hear correctly what she says; or he may misunderstand the meaning of her words.[2]

In this paper, we introduce the possibility of error into a model of strategic information transmission. Information may be lost for two reasons: for the strategic reasons considered by Crawford and Sobel, where the sender *chooses* to withhold information from the receiver, and for the technological reasons just discussed. The possibility of error has a complex effect on the strategic interactions between the agents. The receiver has to take into account that a message he receives may not be what the sender intended, and adjust his decisions accordingly. This affects the sender's incentives, and it turns out that she may try harder to convey information.

The paper is structured as follows. In section 2, we describe the formal details of the model and provide a partial characterization of the equilibrium set. Just as in the CS model, we find that all equilibria are interval partitional; but unlike in CS, coding (the proportion of the message space used by any given set of types) is of critical importance. Section 3 provides a closer examination of the set of equilibria in a special case of the model, analogous to Crawford and Sobel's uniform quadratic case. Via the appropriate coding scheme, we show that one can construct equilibria that induce finitely many, a countable infinity or even an uncountable infinity of actions. Furthermore, for a given number of actions, there is typically a continuum of equilibria that induce that many actions. In section 4 we consider the welfare properties of noise equilibria. Surprisingly, the possibility of error can improve the prospects for communication. We show that for small noise levels there is a simple class of equilibria that are almost always welfare superior to the best CS equilibrium. There exists an optimal noise level for which these equilibria achieve the efficiency bound for general communication devices. Furthermore, for a range of biases introducing any amount of noise can be beneficial. Section 5 examines some extensions of the model, and section 6 concludes.

## 1.1 Related literature

To our knowledge, the idea that noisy communication channels can improve information transmission is first discussed by Myerson [18]. He considers a two-state, three-action cheap talk game; if player 1 is able to send a message to player 2 by means of a carrier pigeon

[1] As an employee of Bell labs, he was primarily concerned with the efficiency of telecommunication systems.

[2] This final source of potential error is particularly important when the words used are vague. Vagueness is a pervasive phenomenon in natural language — consider the use of terms such as "tall", "red", and "good". We return to this theme in the concluding section of the paper.

which arrives only half the time, then communication is possible when it would have been impossible with direct, reliable messages. In the communicative equilibrium, player 1 sends a message only in one of the two states. If the pigeon arrives, player 2 knows that he is in that state; if not, he cannot determine whether the pigeon got lost or was never sent. Effectively then, player 1 reveals information in one state but not in the other, an outcome that is not possible in the absence of noise.

Also related to the current project is the extensive literature on general communication devices (see e.g. Forges [7] and Myerson [17]). Such a device (often thought of as an impartial mediator) receives inputs (messages) from each player, and transmits outputs according to some matrix of transition probabilities. Forges and Myerson show that allowing the players to use these devices can expand the set of equilibrium outcomes in games. Clearly, a communication device could be used to replicate the noise mechanism considered here, or to reproduce the effects of Myerson's unreliable carrier pigeon. But communications devices are much more general than noise mechanisms. Ganguly and Ray [8] provide a partial analysis of the set of equilibria of the CS model that can be achieved if the players have access to a communication device. Their main result concerns devices that are $N$*-simple*: they receive $N$ messages and submit $N$ recommendations. Such devices cannot improve on the $N$-step CS equilibrium if the bias lies below some bound which depends on $N$. Goltsman *et al.* [**?**] investigate the optimal communication devices; we discuss their findings in more detail in section 4.2.1 below.

A paper by Krishna and Morgan [14] shows that allowing multiple rounds of (two-way) communication in the CS framework can also result in equilibria that Pareto dominate those of the original model.[3] They consider a first round of communication, a *meeting*, in which the sender and receiver exchange messages simultaneously, followed by a single transmission from sender to receiver. During the meeting, the sender reveals in which of two elements of the state space the true state lies, and the two agents also send random messages to determine whether the meeting should be deemed a "success" or a "failure". These random messages effectively induce a lottery over outcomes such that neither agent can affect the probability of success or failure. If the meeting was a success, then the sender reveals more information about the true state during the second round of communication; otherwise no more information is revealed. (Clearly this kind of communication could also be replicated using a comminication device: Ganguly and Ray show this formally.) Krishna and Morgan establish the remarkable result that it is almost always possible to construct equilibria in which, relative to the best CS equilibrium, the information gain when the meeting is a success outweighs the information loss when it is a failure, leading to a Pareto improvement. This

[3] Aumann and Hart [1] also examine games with multiple rounds of pre-play communication, and provide a complete characterization of the set of equilibrium outcomes. Since they consider games with a finite set of states, their results do not apply to the CS model.

kind of equilibrium is able to improve on the CS equilibria by leveraging the risk aversion of the sender; in the face of risk about whether or not additional information will be conveyed in the second round, she is willing to give up more information in the first round. In the *uniform quadratic case* most commonly used in applications of the CS model, the welfare results of Krishna and Morgan are similar to our own, although we show that the probability of error can lead to welfare improvements for more extreme values of the sender's bias. But the underlying source of the welfare gain is very different.

Three recent papers introduce different kinds of perturbations into the CS model, and should be discussed next. First, Kartik, Ottaviani and Squintani [13] (henceforth KOS) study strategic information transmission when messages directly affect payoffs, either because the sender faces a cost of lying or receivers are credulous. They show that with an unbounded state space there are fully revealing equilibria. Unlike in their environment, in the noise model analyzed here messages do not have an intrinsic meaning and therefore the notions of deception and language inflation that play an important role in KOS have no content. On the other hand, the issue of coding, i.e. how the message space is used by the various sender types, that plays a crucial role in our analysis does not arise for KOS. Further, in our model there are no fully revealing equilibria, regardless of whether we choose the state space to be bounded or not. One parallel between the two papers is that in both there is a sense in which sender types who separate themselves achieve their ideal points, on average in the KOS model with heterogeneously sophisticated receivers and in the no-noise event in the noise model.

Next, Kartik [12] looks at a perturbation of the CS model in which the sender has an explicit convex cost of misreporting. He finds that that only the most informative CS equilibria can arise as limits of monotonic equilibria as the cost of misreporting converges to zero. Finally, in a closely related paper, Chen [3] modifies the CS model by including a small proportion of behavioral types, *honest* senders and *naive* receivers. Using an additional monotonicity restriction, she shows that there is a unique equilibrium. This equilibrium approaches the maximally-informative CS equilibrium in the limit as the proportions of honest senders and naive receivers converge to zero. In contrast, in the noise model messages have no exogenous meaning and so it is hard to make sense of the notion of honesty. From a technical standpoint, in the noise model we obtain welfare results for a range of strictly positive noise levels, not only in the limit as noise tends to zero; monotonicity of equilibrium is not a significant constraint on the equilibrium set; and, monotonicity in conjunction with the message space being a continuum does not pose existence problems (unlike in Chen's model).

We conclude this review section by mentioning two more variants of the CS model. Olszewski [19] examines a model in which the receiver has private information, there is positive probability that the sender is a behavioral type who always tells the truth rather than being strategic, and the strategic sender prefers to be perceived as the honest type. If the latter concern is sufficiently strong, there is a unique equilibrium which is fully revealing. In Olsweski's model the receiver can ask for more or less information; when the sender cares about the receiver's action, in addition to being perceived as honest, asking for more information may create an incentive for lying. Both of these results are predicated on having behavioral types and messages with intrinsic meanings. In our model, as already discussed, messages acquire meaning only endogenously from the sender's equilibrium strategy, and there are no behavioral types. In Morgan and Stocken's [16] variation of the CS model the receiver is uncertain about the sender's bias. They find equilibria in which there is full separation on a portion of the type space: sender types whose preferences are perfectly aligned with those of the receiver are able to perfectly reveal sufficiently low states of the world. In contrast, in the noise model the possibility of an error in information transmission ensures that the receiver never learns the sender's type for certain, even in equilibria that involve separation on a portion of the type space.

# 2 The Model

## 2.1 Setup

Like Crawford and Sobel, we are interested in the possibility of communication between a privately informed sender, $S$, and a receiver, $R$, who must take some action. The agents' payoffs depend on the sender's information or *type*, $\theta$, and the receiver's action, $a \in \mathbb{R}$. $\theta$ is drawn from a distribution $F(\cdot)$ whose support is the type space $T = [0, 1]$; $F(\cdot)$ is common knowledge and has an everywhere positive density $f(\cdot)$. The sender has a twice continuously differentiable payoff function $U^S(a, \theta, b)$, where $b$ is a parameter measuring her *bias* relative to the receiver. The receiver's twice continuously differentiable payoff function is denoted $U^R(a, \theta) \equiv U^S(a, \theta, 0)$.[4] We assume that for each realization of $\theta$ there exist an action such that $U_1^S(a, \theta, b) = 0$ and an action such that $U_1^R(a, \theta) = 0$; $U_{11}^S(a, \theta, b) < 0 < U_{12}^S(a, \theta, b)$ for all $a, \theta$, and $b$; and $U_{11}^R(a, \theta) < 0 < U_{12}^R(a, \theta)$ for all $a$ and $\theta$. Thus, given the sender's private information, $\theta$, for each player there is a unique ideal action, denoted $a^S(\theta, b)$ and $a^R(\theta)$ for sender and receiver respectively, which maximizes that player's payoff.[5] Note that each player's ideal action is increasing in $\theta$. Finally, we assume that $U_{13}^S(a, \theta, b) > 0$ everywhere, so that an increase in $b$ shifts the sender's preferences further away from the receiver's. Henceforth we disregard the case where sender and receiver have identical

[4] We make this assumption to maintain consistency with the Crawford and Sobel framework. Since we are interested only in values of $b \neq 0$, there is no loss of generality.

[5] That is, $a^S(\theta, b) = \arg\max_a U^S(a, \theta, b)$ and $a^R(\theta) = \arg\max_a U^R(a, \theta)$.

preferences, assuming without loss of generality that $b > 0$, so that $a^R(\theta) < a^S(\theta, b)$ for all $\theta$.

The timing of the game is as follows: the sender observes the value of $\theta$, and then sends a message $m \in M = [\underline{m}, \overline{m}]$; with probability $1 - \epsilon$ the receiver observes the message $m$ sent by the sender; otherwise, with probability $\epsilon$, the receiver observes a message $m'$ that is a draw from the error distribution $G$ on the message space $M$; we assume that he cannot distinguish between received messages that are the result of an error and messages that were sent intentionally. $G$ is independent of the sender's type and of the message sent, and has a density $g$ that is everywhere positive on $[\underline{m}, \overline{m}]$. Finally, the receiver takes some action $a \in \mathbb{R}$. We consider values of $\epsilon \in (0, 1)$, and refer to this game as the *noise model*. In the degenerate case when $\epsilon = 0$, the game collapses to that of Crawford and Sobel — the *CS model*.

## 2.2 Equilibrium

A behavior strategy for the sender $\sigma : T \to \triangle(M)$ specifies what message (or distribution of messages) she sends for each value of $\theta$; and a behavior strategy for the receiver $\rho : M \to \mathbb{R}$ describes what action[6] he chooses for each message he might receive. In a perfect Bayesian equilibrium strategies are optimal given players' beliefs and beliefs are derived from Bayes's rule whenever possible. For a sender of type $\theta$, this means that every message $m$ that she sends must maximize the weighted average of her expected utility if the message is received as intended and her expected utility if there is an error, i.e.

$$\begin{aligned} m &\in \arg\max_{m'} \left( (1-\epsilon) U^S(\rho(m'), \theta, b) + \epsilon \int_{\underline{m}}^{\overline{m}} U^S(\rho(m''), \theta, b) g(m'') \, dm'' \right) \\ &= \arg\max_{m'} U^S(\rho(m'), \theta, b) \end{aligned}$$

(The simplification is possible because the probability of an error, $\epsilon$, and the error distribution, $g(\cdot)$, are independent of the message actually sent.[7]) Now consider the receiver. Let $\mu(\cdot \mid m)$ denote his beliefs about $\theta$ conditional on receiving message $m$. Since $\epsilon > 0$ and $g(\cdot)$ is everywhere positive, Bayes' rule is always well-defined and gives us

$$\mu(\theta \mid m) = \frac{((1-\epsilon)\sigma(m \mid \theta) + \epsilon g(m)) f(\theta)}{\int_0^1 ((1-\epsilon)\sigma(m \mid \theta') + \epsilon g(m)) f(\theta') \, d\theta'}.$$

On receiving message $m$, the receiver chooses the (unique) action which maximizes his ex-

[6] The restriction to pure strategies is without loss of generality: for given beliefs about $\theta$, there is a unique value of $a$ which maximizes $U^R(a, \theta, b)$. This follows immediately from the concavity of $U^R(a, \theta, b)$ in $a$ and the assumption that $a^R(\theta)$ exists for all $\theta$.

[7] We relax the first assumption in section 5.1.

pected utility given these beliefs:

$$\rho(m) = \arg\max_{a'} \int_0^1 U^R(a', \theta) d\mu(\theta \mid m).$$

**Definition 1** *A* perfect Bayesian equilibrium of the noise model *is a strategy for the sender,* $\sigma : T \to \triangle(M)$; *a strategy for the receiver,* $\rho : M \to \mathbb{R}$; *and a set of beliefs for the receiver,* $\mu : M \to \triangle(T)$, *such that*

1. for all $\theta \in T : m \in \arg\max_{m'} U^S(\rho(m'), \theta, b)$, for all $m \in \text{supp}(\sigma(\cdot \mid \theta))$,
2. for all $m \in M : \rho(m) = \arg\max_{a'} \int_0^1 U^R(a', \theta) d\mu(\theta \mid m)$, and
3. $\mu(\theta \mid m) = \frac{((1-\epsilon)\sigma(m|\theta)+\epsilon g(m))f(\theta)}{\int_0^1((1-\epsilon)\sigma(m|\theta')+\epsilon g(m))f(\theta')d\theta'}$.

We use the expressions *noise equilibrium* and *CS equilibrium* to refer to equilibria of the noise model and CS model respectively.

For given parameter values, the set of equilibria is very large, and it is difficult to provide a complete characterization. In this section we derive a number of results about the nature the equilibrium set in the general model just described; in the next section, we provide more results in the context of a specific example (an extension of the well-known uniform-quadratic case of Crawford and Sobel).

We start by introducing some new notation and terminology. Since the sender can only influence the receiver's actions in the no-noise event, it is useful to focus on the receiver's response to the message(s) each type of sender, $\theta$, intends to send, i.e the messages in the support of $\sigma(\cdot \mid \theta)$; let $M_\theta$ denote this set of messages. Then we say that a sender type $\theta$ *induces* action $a$ if there is some $m \in M_\theta$ such that $a = \rho(m)$. We call two equilibria *outcome equivalent* if every sender type induces the same action (or distribution over actions); and two equilibria *essentially outcome equivalent* if at most a countable set of sender types induce different actions. We can now state our first result.

**Proposition 1** *Every noise equilibrium is essentially outcome equivalent to a noise equilibrium in which the type space is partitioned into intervals such that types belonging to the same interval induce the same action and types from different intervals induce distinct actions.*

The proof of all results in this section can be found in the appendix. Proposition 1 implies that almost every type induces precisely one action, and that the set of types which induce any given action is an interval of the type space. The types in this interval may, however, be using different strategies. Proposition 2 shows that it is without loss of generality to confine attention to equilibria in which these types behave identically.

**Proposition 2** *Any noise equilibrium in which* $(\theta_{i-1}, \theta_i)$ *is the interior of the interval of types who induce action* $a_i$ *is outcome equivalent to a noise equilibrium in which*

1. *all these types use the* same *distribution over messages, and*
2. *this distribution is equal to the error distribution,* $G$*, restricted to* $M_i$*, the union of the supports of the distributions* $\sigma(\cdot|\theta)$ *over all* $\theta \in (\theta_{i-1}, \theta_i)$*.*

Notice that each of these sets $M_i$ must be disjoint, since different intervals induce different actions. Our next result shows that we can, without loss of generality, assume that the entire message space is used.

**Proposition 3** *Every noise equilibrium is outcome equivalent to a noise equilibrium in which there are no unused messages.*

There is a close connection between Propositions 1 – 3 above and Theorem 1 of Crawford and Sobel [4], but there are differences between the models that are not immediately obvious from these results. Their Theorem 1 states that every CS equilibrium is outcome equivalent to a CS equilibrium in which types in a given element of the equilibrium partition, $(\theta_{i-1}, \theta_i)$, randomize uniformly over messages in $[\theta_{i-1}, \theta_i]$; but the mixing distribution used is not important, nor is the set of messages used, as long as each partition element uses a distinct set of messages. More precisely, we could construct an outcome equivalent CS equilibrium in which types in $(\theta_{i-1}, \theta_i)$ randomize over messages in some arbitrary set $M_i \subseteq M$, according to some arbitrary distribution $h_i$, as long as the sets used by each interval are disjoint. On the other hand, Proposition 2 describes an equilibrium in which types in a given partition element randomize according to the error distribution restricted to message set $M_i$; in this case (as is evident from the proof) it is crucial that this particular distribution is used. Intuitively, if a different distribution is used, then the receiver's posterior probability of an error, and hence his action, will depend on which message in $M_i$ is observed. Additionally, the size of the set $M_i$ is important, in a sense that is made precise in Proposition 4 below.

Let $\mathcal{M}$ denote a (finite) partition of the message space into intervals $M_1, \ldots, M_N$, and let $m_i$ be the boundary value between intervals $M_{i-1}$ and $M_i$. We say that two partitions $\mathcal{M}$ and $\mathcal{M}'$ are *distinct* if $m_i \neq m_i'$ for some integer $i$ or if they have a different number of elements. A noise equilibrium is *adapted to* $\mathcal{M}$ if there is a partition of the type space into $N$ intervals $T_1, \ldots, T_N$ such that types in $T_i$ randomize over the set $M_i$ according to the error distribution. Denote by $O(\mathcal{M})$ the set of equilibrium outcomes (joint distributions over types and actions) of equilibria that are adapted to $\mathcal{M}$. Then:

**Proposition 4** *If* $\mathcal{M}$ *and* $\mathcal{M}'$ *are distinct, then* $O(\mathcal{M}) \cap O(\mathcal{M}') = \emptyset$*.*

Our next result concerns the relationship between the set of CS equilibria and the set of noise equilibria when the level of noise is low. This proposition requires an additional assumption, Crawford and Sobel's monotonicity condition (M) (see page 1444 of their paper). This condition is satisfied by all standard versions of their model used in the applications, such as the uniform quadratic case; a precise definition can be found in the appendix.

**Proposition 5** *Assume that condition (M) holds and that there exists an $N$-step CS equilibrium. Then there exists $\tilde{\epsilon} > 0$ such that for all noise levels $\epsilon \in (0, \tilde{\epsilon})$ and for any $N$-element partition $\mathcal{M}$ of the message space, there exists an equilibrium that is adapted to $\mathcal{M}$.*

Proposition 4 tells us that that, for fixed $\epsilon$, if $\mathcal{M}$ and $\mathcal{M}'$ are distinct, they cannot produce the same equilibrium outcome. Together with Proposition 5, this implies that near any $N$-step CS equilibrium there is an $N-1$-dimensional set of equilibria all of which induce different equilibrium outcomes.

The final result of this section says that full separation of types is not possible in a noise equilibrium. More precisely, consider a given noise equilibrium in which each type induces precisely one action (by Proposition 1, every noise equilibrium is essentially outcome equivalent to an equilibrium of this kind). Let $\omega : T \to \mathbb{R}$ be the *outcome function*, where $\omega(\theta)$ is the action induced by type $\theta$. Then we say that this equilibrium is *separating* if $\omega$ is one-to-one.

**Proposition 6** *There is no separating noise equilibrium.*

It is worth noting that this result holds even if the state space is not bounded.

# 3 Equilibria in the uniform quadratic case

As noted earlier, it is difficult to give a complete characterization of the equilibrium set. In this section we take a small step in that direction, focusing our attention on the well-known *uniform-quadratic* case introduced by Crawford and Sobel [4]. The remainder of the paper focuses on this case, except where explicitly indicated otherwise. In the uniform quadratic case, the sender's type $\theta$ is drawn from the uniform distribution on the unit interval; the sender's and receiver's utility functions are given by

$$\begin{aligned} U^S &= -(\theta + b - a)^2 \text{ and} \\ U^R &= -(\theta - a)^2 \end{aligned}$$

Notice that the ideal actions of the sender and the receiver are $\theta + b$ and $\theta$ respectively. We also assume that the message space $M = [0, 1]$, and the error distribution $G$ is uniform on $[0, 1]$.

In what follows, we restrict attention to values of bias $0 < b < \frac{1}{2}$, since for larger values no communication is possible and every equilibrium is therefore outcome equivalent to pooling. There are two key differences between the equilibrium sets with and without noise. First, in the noisy case there can be a continuum of equilibrium outcomes of a given number of steps (see section 4.1 below), while in the CS case, Crawford and Sobel show that every $n$-step equilibrium (if any exist) yields the same outcome. Second, we show in sections 4.2 and 4.3 that, as long as the level of noise is high enough, there are equilibria with an infinite and even an uncountable number of steps; in the CS case, on the other hand, every equilibrium has a finite number of steps.

## 3.1 Two-step equilibria

As a starting point, we provide a complete characterization of the set of two-step equilibrium outcomes. Without loss of generality (See Propositions 2 and 3), suppose that types in the first interval, $[0, \theta_1)$ randomize uniformly over messages in $[0, m_1)$, and types in the second interval, $[\theta_1, 1]$ randomize uniformly over messages in $[m_1, 1]$. The actions chosen by the receiver on receiving messages in $[0, m_1)$ and $[m_1, 1]$ are respectively

$$\begin{aligned} a_1 &= \frac{(1-\epsilon)\,\theta_1 \frac{\theta_1}{2} + \epsilon m_1 \frac{1}{2}}{(1-\epsilon)\,\theta_1 + \epsilon m_1} \\ a_2 &= \frac{(1-\epsilon)\,(1-\theta_1)\,\frac{\theta_1+1}{2} + \epsilon\,(1-m_1)\,\frac{1}{2}}{(1-\epsilon)\,(1-\theta_1) + \epsilon\,(1-m_1)} \end{aligned}$$

Equilibrium requires that the sender to be indifferent between $a_1$ and $a_2$ when $\theta = \theta_1$, i.e.

$$\theta_1 + b = \frac{a_1 + a_2}{2}$$

This equation has at most one solution that lies (strictly) between 0 and 1. We denote this solution $\theta_1\,(b, \epsilon, m_1)$.[8] When such a solution exists, of course, it describes a two-step equilibrium. Furthermore, we can show that

$$\frac{\partial \theta_1\,(b, \epsilon, m_1)}{\partial m_1} > 0,$$

i.e. allowing the first step of the equilibrium partition to use a larger proportion of the message space shifts the boundary between the two steps to the right. By computing $\theta_1\,(b, \epsilon, m_1)$ at $m_1 = 0$ and $m_1 = 1$, then, we can obtain lower and upper bounds for $\theta_1$ in any two-step equilibrium:

[8] An explicit expression for $\theta_1\,(b, \epsilon, m_1)$ is given in the appendix.

$$\theta_1(b,\epsilon,0) = \frac{3-4b+4b\epsilon-\sqrt{(3-4b+4b\epsilon)^2-8(1-4b)(1-\epsilon)}}{4(1-\epsilon)}$$

$$\theta_1(b,\epsilon,1) = \frac{1-4b(1-\epsilon)-4\epsilon+\sqrt{1+8b(2b(1-\epsilon)-1)(1-\epsilon)+8\epsilon}}{4(1-\epsilon)}$$

(Notice that when $\epsilon = 0$, both of these expressions are equal to $\frac{1-4b}{2}$, the unique boundary value for a two-step equilibrium in the CS model.) Restricting attention to the relevant range of parameter values ($0 < \epsilon < 1$ and $0 < b < \frac{1}{2}$), we consider two cases:

**Case 1:** $b \in \left(0, \frac{1}{4}\right)$ Here $0 < \theta_1(b,\epsilon,0) < \theta_1(b,\epsilon,1) < 1$. There is a continuum of two-step equilibrium outcomes, corresponding to boundary values in $[\theta_1(b,\epsilon,0), \theta_1(b,\epsilon,1)]$. The size of this set is increasing in $\epsilon$. Figure 1 below illustrates the lower bound (dotted line) and upper bound (solid line) when $b = \frac{1}{10}$.

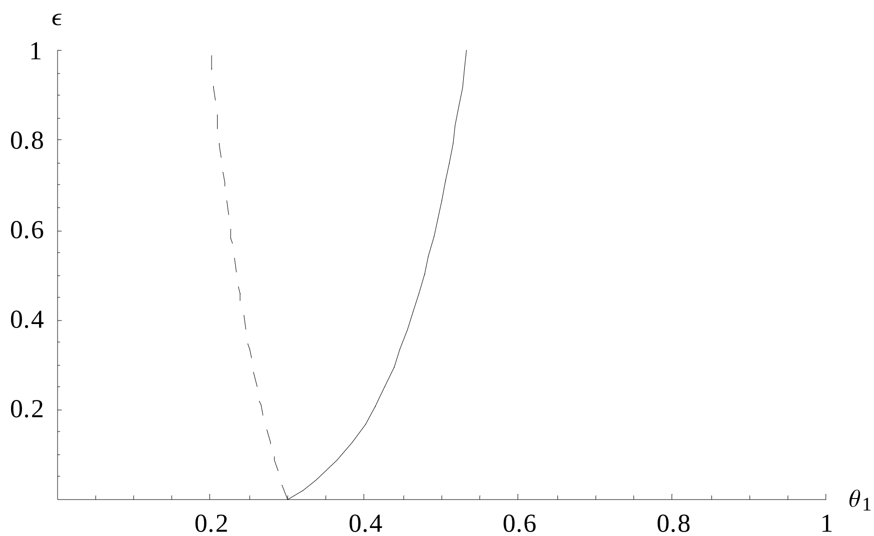


Figure 1: Two-step equilibrium partitions, $b = \frac{1}{10}$.

**Case 2:** $b \in \left[\frac{1}{4}, \frac{1}{2}\right)$ Here $\theta_1(b,\epsilon,0) \leq 0 < \theta_1(b,\epsilon,1) < 1$. Now the lower bound on the set of boundary values is 0, so the set of values that can be attained by varying $m_1$ is $(0, \theta_1(b,\epsilon,1)]$. As before, the size of this set is increasing in $\epsilon$. Note that in this case, the equilibrium construction requires $m_1$ to be above some critical value that is strictly larger than 0.

## 3.2 An infinite partition

We now show that, as long as the level of noise is high enough, there is an equilibrium with an infinite number of steps.[9] A similar construction can be used to prove the existence of an equilibrium with $n$ steps, for any $n > 1$.

We try to construct a equilibrium partition of the following form:

$$\{\{0\}, \ldots, [\theta_{-3}, \theta_{-2}), [\theta_{-2}, \theta_{-1}), [\theta_{-1}, 1]\},$$

where $\theta_{i-1} < \theta_i$ and $0 < \theta_i < 1$ for $i \leq -1$, and $\lim_{i \to -\infty} \theta_i = 0$ (so the set does indeed partition $[0, 1]$). Suppose that type $\theta = 0$ sends message $m = 0$; all types $\theta \in [\theta_{i-1}, \theta_i)$ $(i \leq -1)$ randomize uniformly over messages in $[\zeta\theta_{i-1}, \zeta\theta_i)$; and types $\theta \in [\theta_{-1}, 1]$ randomize uniformly over messages in $[\zeta\theta_{-1}, 1]$ ($\zeta$ is a constant whose value will be determined later). Consider the receiver's best response to this strategy of the sender. Conditional on receiving a message $m \in [\zeta\theta_{i-1}, \zeta\theta_i)$, the receiver's posterior belief that the message was received in error is given by

$$\eta = \frac{\epsilon\zeta(\theta_i - \theta_{i-1})}{(1-\epsilon)(\theta_i - \theta_{i-1}) + \epsilon\zeta(\theta_i - \theta_{i-1})} = \frac{\epsilon\zeta}{(1-\epsilon) + \epsilon\zeta}.$$

Thus the receiver's optimal action, $a_i$, solves

$$\max_a (1-\eta) \int_{\theta_{i-1}}^{\theta_i} -(\theta - a)^2 \frac{1}{\theta_i - \theta_{i-1}} d\theta + \eta \int_0^1 -(\theta - a)^2 d\theta$$
$$\Rightarrow \quad a_i = (1-\eta)\frac{\theta_i + \theta_{i-1}}{2} + \eta\frac{1}{2} \text{ (for } i \leq -1).$$

On receiving a message $m \in [\zeta\theta_{-1}, 1]$, on the other hand, it is easy to verify that the receiver's optimal response is to choose action

$$a_0 = \frac{(1-\epsilon)(1-\theta_{-1})\frac{\theta_{-1}+1}{2} + \epsilon(1-\zeta\theta_{-1})\frac{1}{2}}{(1-\epsilon)(1-\theta_{-1}) + \epsilon(1-\zeta\theta_{-1})}.$$

Turning now to the sender's strategy, we need each sender boundary type $\theta_i$ to be indifferent between inducing action $a_i$ and action $a_{i+1}$, i.e.

$$\theta_i + b = \frac{a_i + a_{i+1}}{2} \text{ (for } i \leq -1).$$

[9] A recent paper by Gordon [10] also demonstrates the existence of equilibria with an infinite number of steps in a framework that is based on the CS model. Specifically, his Theorem 4 states that such equilibria exist as long as preference satisfy a *moderate audience* condition, which says that the lowest sender type has a negative bias while the highest sender type has a positive bias.

Notice that each of these *indifference conditions* (except for $\theta_{-1}$) involves $\theta_{i+1}$, $\theta_i$ and $\theta_{i-1}$. Solving for $\theta_{i+1}$, we obtain the second-order difference equation

$$\theta_{i+1} = \frac{2+2\eta}{1-\eta}\theta_i - \theta_{i-1} + \frac{4b-2\eta}{1-\eta} \text{ (for } i \leq -2).$$

We have said nothing yet about the value of $\zeta$. Let

$$\zeta = \frac{2b(1-\epsilon)}{\epsilon(1-2b)},$$

so that $\eta = 2b$ (Notice that $\zeta > 0$ as long as $0 < b < \frac{1}{2}$). Then a solution to this difference equation is

$$\theta_{i-1} = \theta_{-1}\left(\frac{1-\sqrt{2b}}{1+\sqrt{2b}}\right)^{-i} \quad (i = \ldots, -2, -1, 0).$$

As long as we choose a value of $\theta_{-1} \in (0,1)$, we have $\theta_{i-1} < \theta_i$ and $0 < \theta_i < 1$ for $i \leq -1$, and $\lim_{i\to-\infty}\theta_i = 0$, as required.

The remaining indifference condition, however, fixes the value of $\theta_{-1}$:

$$\begin{aligned} \theta_{-1} + b &= \frac{a_{-1}+a_0}{2} \\ \Rightarrow \theta_{-1} &= \frac{1-\sqrt{2b}}{1+\sqrt{\epsilon}} \end{aligned}$$

For this construction to work, we need to make sure that the sender's strategy described at the beginning of this section is well-defined, i.e. there are some messages left over for the final interval of sender types to send. This requires

$$\begin{aligned} \zeta\theta_{-1} &\leq 1 \\ \Rightarrow \epsilon &\geq \frac{2b}{\left(1+\sqrt{2b}\right)^2} \end{aligned}$$

The existence of the kind of equilibrium described above, then, with an infinite number of steps, relies on there being "enough" noise. The level of noise required depends on the size of the sender's bias. Figure 2 belows shows this threshold value of $\epsilon$ for different values of $b$.

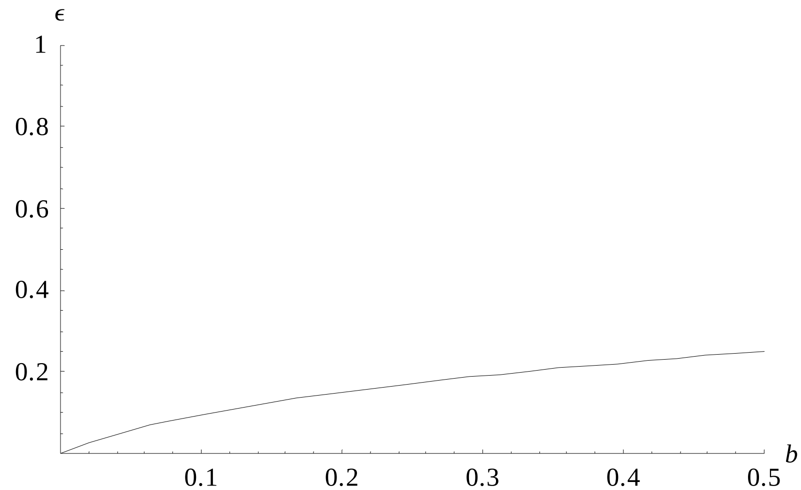


Figure 2: Threshold value of $\epsilon$ as a function of $b$.

Finally, we note that a very similar construction can be used to demonstrate the existence of an $N$-step equilibrium for any *finite* $N \geq 3$ (section 3.1 above deals with the case where $N = 2$) To see how, consider the $N$-step partition

$$\{[0, \theta_{1-N}), [\theta_{1-N}, \theta_{2-N}), \ldots, [\theta_{-2}, \theta_{-1}), [\theta_{-1}, 1]\}.$$

Suppose that the sender adopts the same strategy as before, with types $\theta \in [\theta_{i-1}, \theta_i)$ ($i \leq -1$) randomizing uniformly over messages in $[\zeta\theta_{i-1}, \zeta\theta_i)$ and types $\theta \in [\theta_{-1}, 1]$ randomizing uniformly over messages in $[\zeta\theta_{-1}, 1]$. The indifference conditions for boundary types $\theta_{1-N}, \ldots, \theta_{-2}$ yield the same second-order difference equation as above. Solving this equation, with the boundary condition $\theta_{-N} = 0$ and treating $\theta_{-1}$ as a parameter, we can compute $a_{-1}$ (the action corresponding to the penultimate step) as a function of $\theta_{-1}$. For any given $\theta_{-1}$, the value of $a_{-1}$ will be lower than in the infinite case, since the steps are more spaced out. The final indifference condition, therefore, gives a lower value of $\theta_{-1}$, so the threshold value of $\epsilon$ required for this construction to work is strictly lower than before.

## 3.3 Uncountable partitions

The construction of the previous sections shows that, as long as the level of noise is on or above some threshold value, there is a noise equilibrium with a countably infinite number of steps. If $\epsilon$ is strictly larger than this value, we can find a noise equilibrium with an uncountable number of steps. To prove this we use a construction similar to that described in the previous section, except that at the left hand end of the type space (i.e. for low values of $\theta$) the sender adopts a fully-revealing strategy, with every type sending a distinct message.

For the sake of exegesis, however, we here provide a simpler construction which works as long as $\epsilon > 2b$ (note that the simplification comes at the expense of a tighter constraint). Consider the following partition:

$$\left\{ \{\{\theta\}\}_{\theta\in[0,\theta^*]}, \ldots, (\theta_{-3}, \theta_{-2}], (\theta_{-2}, \theta_{-1}], (\theta_{-1}, 1] \right\},$$

where $\theta_{i-1} < \theta_i$ and $0 < \theta^* < \theta_i < \theta_0 = 1$ for $i \leq -1$, and $\lim_{i\to-\infty} \theta_i = \theta^*$. Suppose the sender adopts the following strategy:

- if $\theta \in [0, \theta^*]$, send message $m = s(\theta)$ where $s(\cdot)$ is a strictly increasing differentiable function with $s(0) = 0$;
- if $\theta \in (\theta_{i-1}, \theta_i]$ $(i \leq 0)$, randomize uniformly over messages in $(\zeta(\theta_{i-1} - \theta^*) + s(\theta^*),$ $\zeta(\theta_i - \theta^*) + s(\theta^*)]$, where $\zeta(1 - \theta^*) + s(\theta^*) = 1$.

(Note that we use intervals that are open on the left rather than on the right here merely to simplify notation.) Now consider the receiver's best response. Suppose he receives a message $m \in [0, s(\theta^*)]$; the distribution of sent messages in this continuous portion of the message space is given by $s^{-1}(m)$ with density

$$\frac{1}{s'(s^{-1}(m))}.$$

Hence, conditional on receiving a message $m \in [0, s(\theta^*)]$, the posterior probability that $m$ was received by error is given by[10]

$$\mu(m) \equiv \frac{\epsilon}{\epsilon + \frac{(1-\epsilon)}{s'(s^{-1}(m))}}.$$

The receiver then chooses the action $a$ that maximizes

$$-(1 - \mu(m))\left(a - s^{-1}(m)\right)^2 - \mu(m)\int_0^1 (a - \theta)^2 \, d\theta.$$

[10] Let $m \in [m', m''] \subset [0, s(\theta^*)]$ and recall that the error distribution is $G(m) = m$. Then the probability that a message was received in error, $E$, conditional on knowing that the message is in $[m', m'']$ equals

$$\begin{aligned}
P(E|[m', m'']) &= \frac{P([m', m'']|E))\epsilon}{P([m', m'']|E)\epsilon + P([m', m'']|\neg E)(1-\epsilon)} \\
&= \frac{(m'' - m')\epsilon}{(m'' - m')\epsilon + (s^{-1}(m'') - s^{-1}(m'))(1-\epsilon)} \\
&= \frac{\epsilon}{\epsilon + \frac{s^{-1}(m'') - s^{-1}(m')}{m'' - m'}(1-\epsilon)}
\end{aligned}$$

Now consider the limit as $m'' - m' \to 0$

The maximum is achieved at

$$a_m = (1 - \mu(m))\, s^{-1}(m) + \mu(m)\, \frac{1}{2}.$$

Let $\theta = s^{-1}(m)$. Clearly, if the receiver's optimal response matches the sender's ideal point $(\theta + b)$, the sender will have no incentive to deviate from the specified strategy (for $\theta \in [0, \theta^*]$). Formally:

$$\begin{aligned} & a_m = \theta + b \\ \Rightarrow\ & (1 - \mu(m))\, \theta + \mu(m)\, \frac{1}{2} = \theta + b \\ \Rightarrow\ & \mu(m) = \frac{2b}{1 - 2\theta} \end{aligned}$$

which can be solved for

$$s'(\theta) = \frac{2b(1 - \epsilon)}{\epsilon(1 - 2b - 2\theta)}.$$

Using the boundary condition $s(0) = 0$, we obtain the sender's strategy for types $\theta \in [0, \theta^*]$:

$$s(\theta) = -\frac{b(1 - \epsilon)}{\epsilon} \ln\left(\frac{1 - 2b - 2\theta}{1 - 2b}\right).$$

Next, suppose that the message received is in the interval $(\zeta\theta_{i-1} + s(\theta^*), \zeta\theta_i + s(\theta^*)]$ $(i \leq 0)$; then the receiver's optimal action is given by

$$a_i = (1 - \eta)\, \frac{\theta_i + \theta_{i-1}}{2} + \eta \frac{1}{2},$$

where $\eta$ is as defined in the previous section. As before, it follows that boundary types must satisfy the difference equation

$$\theta_{i+1} = \frac{2 + 2\eta}{1 - \eta}\theta_i - \theta_{i-1} + \frac{4b - 2\eta}{1 - \eta} \text{ (for } i \leq -1).$$

We need a solution of this difference equation that converges to $\theta^*$ — this ensures that the required indifference condition for the boundary type at $\theta^*$ will be satisfied. Hence, $\theta^*$ must satisfy

$$\theta^* = \frac{2+2\eta}{1-\eta}\theta^* - \theta^* + \frac{4b-2\eta}{1-\eta}$$
$$\Rightarrow \theta^* = \frac{\eta-2b}{2\eta}$$
$$\Rightarrow \theta^* = \frac{1}{2} - \frac{b\left(1-\epsilon\left(1-\zeta\right)\right)}{\epsilon\zeta}$$

Solving for $\zeta$, we obtain

$$\zeta = \frac{2b\left(1-\epsilon\right)}{\epsilon\left(1-2\theta^*-2b\right)}, \text{ so}$$
$$\eta = \frac{2b}{1-2\theta^*}.$$

The difference equation becomes:

$$\theta_{i+1} - \frac{2-4\theta^*+4b}{1-2\theta^*-2b}\theta_i + \theta_{i-1} = -\frac{8\theta^* b}{1-2\theta^*-2b}.$$

With the boundary constraint $\theta_0 = 1$, the solution is

$$\theta_i = \left(1-\theta^*\right)\left(\frac{1-2\theta^*+2b-\sqrt{4b\left(2-4\theta^*\right)}}{1-2\theta^*-2b}\right)^{-i} + \theta^*.$$

Finally, recall that we require that

$$\zeta\left(1-\theta^*\right) + s\left(\theta^*\right) = 1,$$

so all messages are used in equilibrium. Can we find a $\theta^*$ which solves this equation? Notice that $s(0) = 0$, and $s'\left(\theta^*\right)$ is strictly increasing for $\theta^* \in \left[0, \frac{1}{2}-b\right)$ with $\lim_{\theta^*\to\frac{1}{2}-b} = \infty$. Further, $\zeta\left(1-\theta^*\right)$ is increasing in $\theta^*$ (as long as $b < \frac{1}{2}$). So, by continuity, we can find a solution to this equation if and only if $\zeta\left(1-\theta^*\right) < 1$ when $\theta^* = 0$. This implies that $\epsilon > 2b$.

As an example, suppose $b = \frac{1}{5}$ and $\epsilon = \frac{4}{5}$. then we have an equilibrium with $\theta^* = 0.259$, $s\left(\theta^*\right) = 0.099$, and $\zeta = 1.215$.

# 4 Welfare results

The results of the previous section suggest a sense in which, if the information transmission process is noisy, more communication is possible — we found noise equilibria in which the sender's messages partition the state space more evenly and into more elements than is possible in any CS equilibrium; furthermore, introducing noise allows us to construct communicative equilibria for values of $b$ that are so high that the only CS equilibrium is totally uninformative (specifically, $b \in \left[\frac{1}{4}, \frac{1}{2}\right)$ in the uniform quadratic case).

We call any changes in the agents' utility resulting from changes in the equilibrium partition the *strategic effect* of noise. What is the source of this effect? Recall that in a CS equilibrium communication is imperfect because the sender and the receiver do not agree on what action should be chosen for any type. In the presence of noise, the receiver has to take into account the possibility that a given message was received in error; his expectation of the sender's type is a weighted average of the expectation given that the message was transmitted faithfully and the expectation given that there was an error. Compared with the noiseless case, the receiver's expectations are distorted towards the *ex ante* mean. In particular, the meaning of a message that signals a low type is distorted upwards; this implies that the receiver's action will also be distorted upwards, and hence closer to the ideal action of the sender (given positive bias). For low types, then, noise brings the effective preferences of the sender and receiver into closer alignment. Even though the opposite is true for high types, this allows us to construct more evenly-spaced equilibrium partitions.

We are some way from concluding that noise facilitates information transmission, however. There are two negative effects of noise, which mitigate the strategic effect. First, when errors actually occur, there is a clear loss of information (the *direct effect*); second, since the receiver does not observe whether or not a given message was sent in error, he has to trade off the losses in each contingency (the *distortion effect*).

To analyze the trade-off between these three effects, we need a precise measure of the informativeness of an equilibrium: we follow Crawford and Sobel in using the (*ex ante*) expected utility of the receiver for this purpose. In the uniform quadratic case, which we continue to focus on in this section, this is equal to the negative of the residual variance of $\theta$ that the receiver expects to face after receiving his message. Further, it can be shown that (in equilibrium) $EU^S = EU^R - b^2$, so this measure also gives us a Pareto ranking of equilibria — one equilibrium is more informative than another if and only if it Pareto dominates it. In example 1 below, we show how much of the receiver's change in utility once noise is introduced is due to each of the three effects described above.

Two questions naturally arise regarding the welfare properties of noise equilibria. First, what does the most informative noise equilibrium look like? And second, does noise increase or reduce informativeness? We consider the second question first, and prove two key results: (1) a small amount of noise is (almost) always a good thing; and (2) if the bias is large, any amount of noise is a good thing. The first result is expressed formally in Proposition 7.

## 4.1 The welfare effects of noise

### 4.1.1 Low noise

**Proposition 7** *If $b < \frac{1}{2}$ and $b \neq \frac{1}{2N^2}$ for some integer $N > 1$, there exists an $\overline{\epsilon} > 0$ such that for all $\epsilon \in (0, \overline{\epsilon})$ there is an $\epsilon$-noise equilibrium that is Pareto superior to all CS equilibria.*

The proof of this proposition can be found in the appendix. We construct an $\epsilon$-noise equilibrium and show that it is more informative than every CS equilibrium for small values of $\epsilon$. To see how the construction works, consider the following example.

**Example 1**

Suppose that $b = \frac{1}{10}$. We compare the receiver's expected utility in the Pareto optimal CS equilibrium and a three-step noise equilibrium, with noise $\epsilon = \frac{1}{126}$.

*CS equilibrium*

The Pareto optimal CS equilibrium has two steps, with partition elements $\left[0, \frac{3}{10}\right)$ and $\left[\frac{3}{10}, 1\right]$, and resulting $EU^R = -\frac{37}{1200} = -0.0308$

*Three-step noise-equilibrium*

Consider the partition $\left\{\left[0, \frac{1}{25}\right), \left[\frac{1}{25}, \frac{8}{25}\right), \left[\frac{8}{25}, 1\right]\right\}$. Suppose that the sender obeys the following strategy:

$$\begin{aligned}
\text{If } \theta &\in \left[0, \frac{1}{25}\right), \text{ randomize uniformly on } [0,1] \setminus \{m_2, m_3\} \\
\text{If } \theta &\in \left[\frac{1}{25}, \frac{8}{25}\right), \text{ send message } m_2 \\
\text{If } \theta &\in \left[\frac{8}{25}, 1\right], \text{ send message } m_3
\end{aligned}$$

Notice that the sender adopts a *front-loading* strategy, using up almost all of the messages in the first partition element.[11] With probability one, then, if there is an error in message transmission, the message received coincides with one of the messages sent by that first partition element. The receiver's best response is to choose actions according to the following

[11] The reader may recall from the characterization of two-step equilibrium in section 3.1 above that front-loading maximizes the size of the first partition element. As long as the level of noise is small, the first partition element is the smallest; increasing its size thus makes the equilibrium more informative, *ceteris paribus*.

strategy:

$$\begin{aligned}
\text{If } m &\in [0,1] \setminus \{m_2, m_3\} \text{ is received, choose } a_1 = \frac{1}{10} \\
\text{If } m &= m_2 \text{ is received, choose } a_2 = \frac{9}{50} \\
\text{If } m &= m_3 \text{ is received, choose } a_3 = \frac{33}{50}
\end{aligned}$$

In each case, the action chosen is equal to the receiver's expectation of $\theta$ given his information. Notice that, for the second and third partitions elements, this is simply the midpoint of the interval. This is because messages $m_2$ and $m_3$ are sent by error with probability zero, so the receiver can be certain that the sender's type is in the relevant interval. This eliminates the distortion effect except for the first (and smallest) partition element.

To check that we have an equilibrium, we need to verify that the sender's strategy is also a best response. This amounts to checking that the boundary types $\theta_1 = \frac{1}{25}$ and $\theta_2 = \frac{8}{25}$ satisfy the indifference conditions:

$$\begin{aligned}
\theta_1 &: \frac{1}{25} = \frac{a_1 + a_2}{2} - b = \frac{\frac{1}{10} + \frac{9}{50}}{2} - \frac{1}{10} = \frac{1}{25} \qquad \checkmark \\
\theta_2 &: \frac{8}{25} = \frac{a_2 + a_3}{2} - b = \frac{\frac{9}{50} + \frac{33}{50}}{2} - \frac{1}{10} = \frac{8}{25} \qquad \checkmark
\end{aligned}$$

The resulting expected utility for the receiver is $EU^R = -\frac{36}{1200}$ (see the appendix for the calculation). As we can see, the additional information conveyed by the sender more than compensates for the loss of information through noise, resulting in a Pareto improvement compared to the CS equilibrium.

Figure 3 provides a graphical illustration of these equilibria. The boundary points are shown above the unit interval, and the actions chosen in each case are given below.

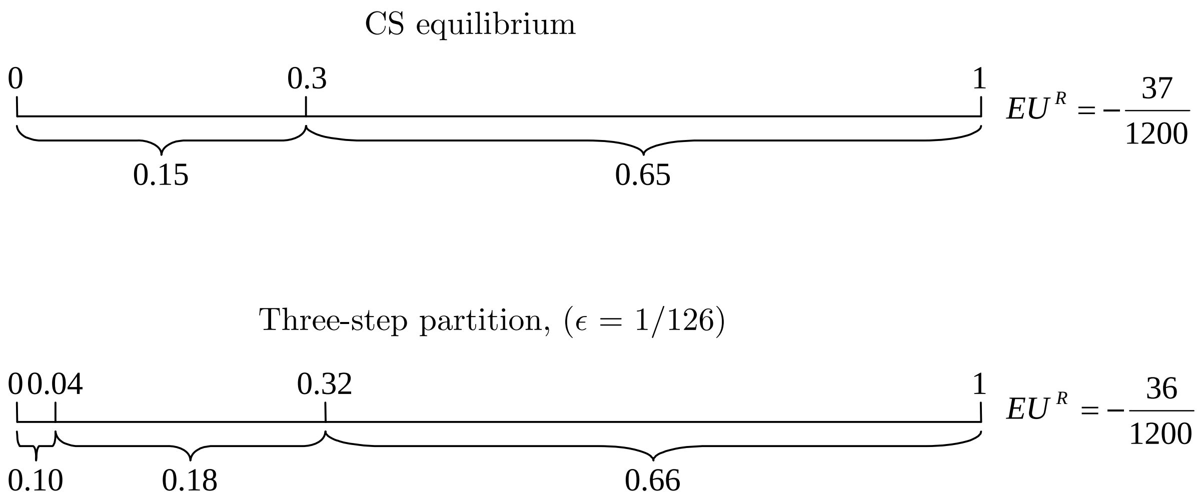


Figure 3: Equilibria with $b = \frac{1}{10}$

How much of this overall change in $EU^R$ is due to the three effects discussed above? To calculate the strategic effect, we compute the receiver's expected utility if there were no noise, but his information partition is the same as in the noise equilibrium $\left\{ \left[0, \frac{1}{25}\right), \left[\frac{1}{25}, \frac{8}{25}\right), \left[\frac{8}{25}, 1\right] \right\}$ (of course, this is not an equilibrium). For the direct effect, we take this value from his utility if he had this information partition in the no-noise event, and no information in the noise event (so we are effectively assuming that he knows whether a given message was sent in error or not). The remaining change is due to the distortionary effect, which isolates the utility loss resulting because the receiver cannot in fact distinguish messages sent in error from correct ones. Table 1 shows the size of each of these effects.

Table 1: Decomposition of change in $EU^R$ when noise is introduced

| | | | | | | |
|---|---|---|---|---|---|---|
| $-0.0308$ | $\longrightarrow$ | $-0.0280$ | $\longrightarrow$ | $-0.0285$ | $\longrightarrow$ | $-0.0300$ |
| | strategic effect | | direct effect | | distortionary effect | |
| | (+0.0028) | | ($-0.0004$) | | ($-0.0015$) | |

The threshold level of noise, $\bar{\epsilon}$, below which the front-loading equilibrium generates a Pareto improvement over the best CS equilibrium is shown in the diagram below. High values of $b$ are omitted for the sake of clarity; $\bar{\epsilon}$ rises from 0 to 1 as $b$ goes from $\frac{1}{8}$ to $\frac{1}{4}$, and $\bar{\epsilon} = 1$ for $b \in \left[\frac{1}{4}, \frac{1}{2}\right)$ (see also Proposition 8 below). Clearly, $\bar{\epsilon}$ is a non-monotonic function of $b$. It turns out that whenever $b = \frac{1}{2N^2}$ ($N = 2, 3, \ldots$), the most informative CS equilibrium is Pareto optimal in a very general class of communication protocols[12], which

[12] This result is implied by Lemma 1 of Goltsman *et al.* [9], discussed in more detail in section 4.2.1 below.

includes noisy talk. When the bias is equal to these values, then, noise cannot generate a Pareto improvement, and $\bar{\epsilon} = 0$. For other values of $b$, however, a small amount of noise can be beneficial. The further $b$ is from these critical values, the more potential there is for a Pareto improvement, and hence the larger the value of $\bar{\epsilon}$. The peaks of the graph are at $b = \frac{1}{2N(N-1)}$ $(N = 2, 3, \ldots)$.

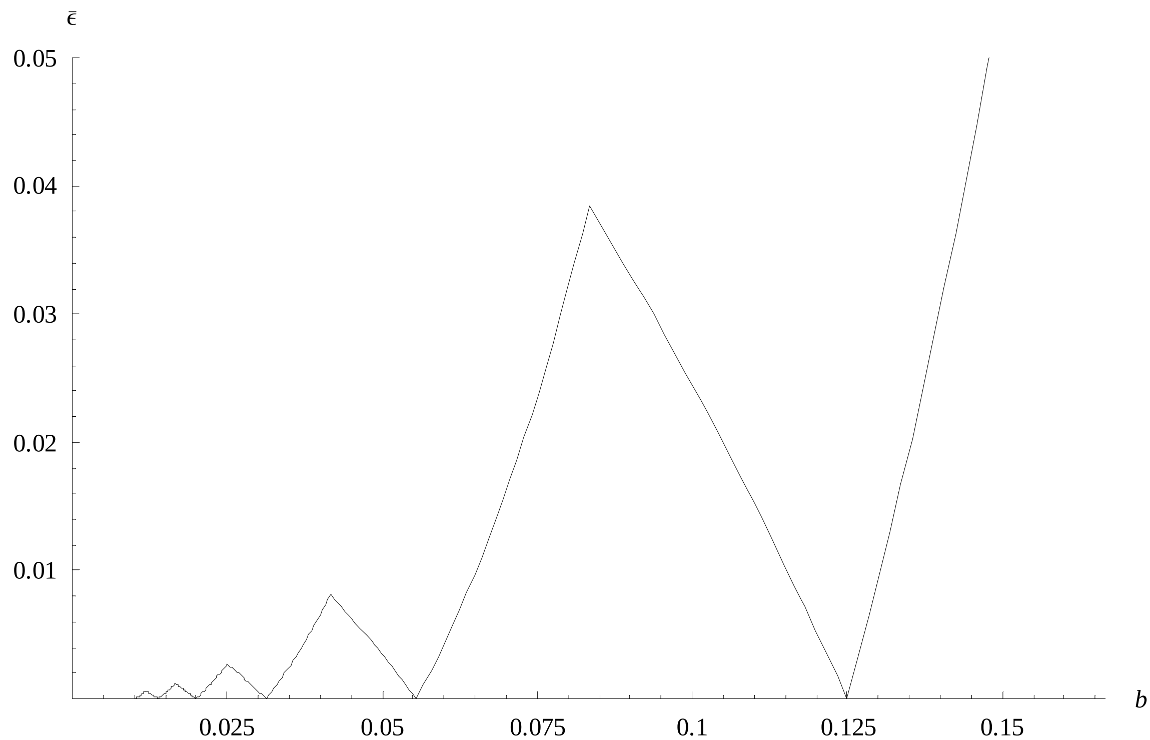


Figure 4: Maximum level of noise for a Pareto improvement

As already mentioned, the technique used to construct the noise equilibrium in example 1 is to have the sender employ a front-loading strategy, where the first partition element uses almost all of the message space (a generalization of this construction is used in the proof of Proposition 7 in the appendix). This strategy is effective because if the sender's type lies in any of the *other* partition elements, and her message is relayed faithfully, the receiver can be certain that there was no error. An identical result could be achieved in a framework where messages are simply lost (rather than garbled) with small probability. To see how, suppose that types in the lowest partition element do not send any messages, and all types in the other partition elements send distinct message. If the receiver observes a message, he can be certain which partition element it came from, just as if he receives message $m_2$ or $m_3$ in example 1; on the other hand, if he receives no message, he has to balance the probability that no message was sent (and therefore the sender's type is in the lowest partition element) against the probability that a message was sent but it got lost (and so her type is is higher).

**The role of risk aversion** We have just seen that the introduction of a small amount of noise into the information transmission process can result in welfare improvements. The source of this welfare gain is the strategic effect: the presence of noise induces the sender to reveal more information than she otherwise would. When $\epsilon$ is low, this effect dominates the direct effect of lost messages. This result is perhaps even more surprising when we consider that our agents are risk averse: utility is a concave function of distance from their ideal actions. In fact, risk aversion helps as well as hinders. Consider the position of the receiver, faced with a message that may have been sent in error. To minimize his expected loss, he adjusts his action toward the *ex ante* expectation of $\theta$. The size of this adjustment depends, of course, on the amount of noise, but also on the degree of risk aversion. Very risk averse agents are more concerned about the small probability of being far from the ideal action, and will therefore make a larger adjustment. This implies that the receiver's actions are less responsive to the different messages sent by the sender, who thus has less incentive to exaggerate, so that more informative partitions are possible.

To examine the effects of risk aversion more precisely, we analyze a simple example with the following class of utility function:

$$\begin{aligned} U^S &= -\left|\theta + b - a\right|^k \text{ and} \\ U^R &= -\left|\theta - a\right|^k \end{aligned}$$

where $k \geqslant 1$. Notice that $k$ is a measure of the curvature of the utility functions, and so can be thought of as representing the degree of risk aversion of the agents. When $k = 2$, we have the quadratic case considered above. Using the same parameter values as in example 1 ($b = \frac{1}{10}$, $\epsilon = \frac{1}{126}$), we compute noise equilibria for $k = 1, 2, 3, 4$. In each case we use the same construction as in section 3.1 above, where the sender types in the first element of the partition send (almost) all of the messages. Figure 5 summarizes our results. For $k = 1$, the (risk-neutral) linear loss function, the equilibrium partition is almost identical to the CS equilibrium partition ($\{[0, 0.3), [0.3, 1]\}$); as the degree of risk aversion increases, the equilibrium partition becomes much more informative.

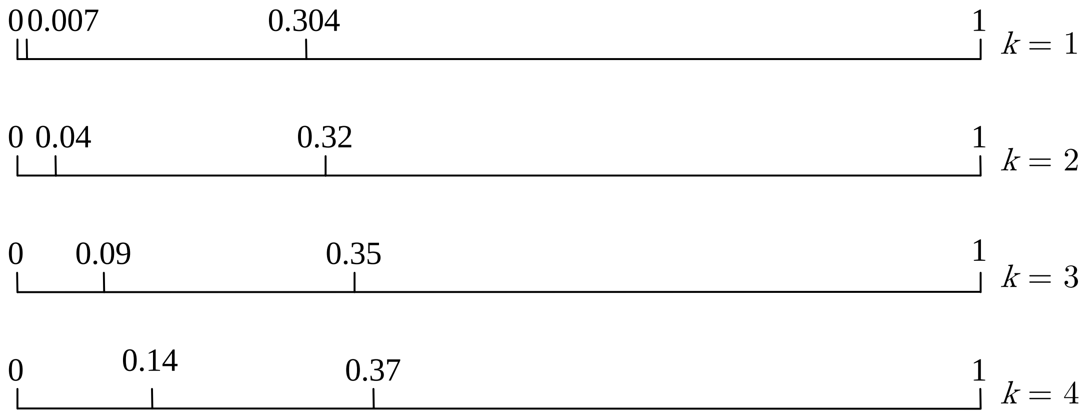


Figure 5: The effect of risk aversion

This example demonstrates that noise can generate a Pareto improvement even in the risk neutral case (the noise equilibrium described above yields an expected utility for the receiver of $-0.124$, as opposed to $-0.145$ in the most informative CS equilibrium). This contrasts with the results of Krishna and Morgan [14]: they construct equilibria in which multiple rounds of communication can be beneficial by leveraging the risk aversion of the sender.

#### 4.1.2 High bias

Returning to the case of quadratic preference, our next result states that any amount of noise can be beneficial as long as the bias is large enough.

**Proposition 8** *For all $b \in \left[\frac{1}{4}, \frac{1}{2}\right)$ and all $\epsilon \in (0,1)$ there is an $\epsilon$-noise equilibrium that is Pareto superior to all CS equilibria.*

To prove this result, we refer back to the characterization of the set of two-step equilibria in section 3.1 above. There we show that if we set $m_1$ equal to 1 (i.e. the sender follows a front-loading strategy), then a two step equilibrium exists for all $b < \frac{1}{2}$ and $\epsilon \in (0,1)$. On the other hand, for $b \in \left[\frac{1}{4}, \frac{1}{2}\right)$ the unique CS equilibrium outcome is completely uninformative. Since the receiver is strictly better off with some information rather than none, the result follows.

Proposition 8 extends straightforwardly to the family of utility functions considered in section 4.1.1 above. In the more general framework of section 2, the link between sender's and receiver's utility is broken and equilibria cannot always be Pareto ranked. The finding that noise enables communication when it would otherwise not have been possible, however, seems fairly robust. Suppose that Crawford and Sobel's monotonicity condition (M) holds[13],

[13] See page 1444 of Crawford and Sobel [4], or the proof of Proposition 5 in the appendix below.

and let $b^*$ be the lowest bias level for which the unique CS equilibrium outcome is pooling (i.e. $b^*$ is the level of bias that is just too high for communication to be possible). Then we can show that, for this bias level, there exists a two-step noise equilibrium for any level of noise.[14] This equilibrium is better for the receiver, but not necessarily for the sender, than the CS equilibrium. A related result is obtained by Austen-Smith [2] in a rather different context. In his model, the sender may or may not know the value of her type; the receiver is unable to determine whether she is informed or not (there is *receiver uncertainty*), and the sender is allowed to send a message if and only if she is informed. He shows that, for a given set-up, if there is an informative equilibrium of the CS model then there is an informative equilibrium of the receiver-uncertainty model; but there is a range of values of sender bias for which there is an informative equilibrium of the receiver-uncertainty model only. In this sense, receiver uncertainty, like noise, facilitates communication. From a formal standpoint, the equilibrium construction used by Austen-Smith to prove this result resembles a two-step front-loading noise equilibrium of our model: Informed sender types in the first partition element pool with uninformed sender types, sending no message, in the same way that in a front-loading noise equilibrium sender types in the first partition element can be thought of as pooling with types who suffered from the error event; on the other hand, types in the second partition element guarantee self-identification by sending *some* message (for Austen-Smith) or by sender a specific message that is received with zero probability in the error event (in our noise equilibrium).

## 4.2 Optimal noise equilibria

Propositions 7 and 8 describe circumstances under which we can find noise equilibria that Pareto dominate the best CS equilibrium. But we would also like to know, for given parameter values ($b$ and $\epsilon$), what is the optimal noise equilibrium. We are able to provide only a very partial answer to this question. Specifically, for given $b$, we are able to find the optimal noise equilibrium if $\epsilon$ is a choice variable (section 4.2.1); and we believe that the optimal two-step noise equilibrium always involves front-loading (section 4.2.2). For arbitrary $b$ and arbitrary $\epsilon$, however, we do not know what optimal equilibria look like. One problem is that, unlike in the CS model[15], equilibria with more steps do not necessarily Pareto dominate equilibria with fewer steps. First, noise equilibrium partitions with more steps may nevertheless divide the state space less evenly than equilibrium partitions with fewer steps, and hence provide less information (consider the two-step and three-step partitions in Figure 5 below). Second, the way that the message space is used by the various partition elements (the *coding* of messages) is also important: in general, the more messages that are used by

[14] Details are available from the authors upon request.

[15] Again, restricting attention to the uniform quadratic case.

a given partition element, the more distortion is created, since it is harder to distinguish whether such messages were sent by error or not. For a given partition, then, a particular coding will minimize the distortion effect (perhaps having all of the messages are sent by the smallest partition element). But changing the coding changes the equilibrium partition, and there might be a trade-off between the kind of coding which minimizes the distortion effect and the kind of coding which generates the finest partition (i.e. maximizes the strategic effect).

Example 2 illustrates the point that noise equilibria with more steps are not necessarily better.

**Example 2**

Suppose that $b = \frac{1}{8}$ and $\epsilon = \frac{1}{5}$.

*Two-step equilibrium with front loading of messages*

Consider the partition $\left\{ \left[0, \frac{3}{8}\right), \left[\frac{3}{8}, 1\right] \right\}$. Suppose that the sender obeys the following strategy:

$$\begin{aligned} \text{If } \theta &\in \left[0, \frac{3}{8}\right), \text{ randomize uniformly on } [0,1] \setminus \{1\} \\ \text{If } \theta &\in \left[\frac{3}{8}, 1\right], \text{ send message } 1 \end{aligned}$$

Then the receiver's best response is:

$$\begin{aligned} \text{If } m &\in [0,1] \setminus \{1\} \text{ is received, choose } a_1 = \frac{5}{16} \\ \text{If } m &= 1 \text{ is received, choose } a_2 = \frac{11}{16} \end{aligned}$$

It is easy to check that the indifference condition for the sender type $\theta = \frac{3}{8}$ is satisfied, and we have a noise equilibrium. The receiver's expected utility is $EU_R = -0.048$.

*Three-step equilibrium with proportional coding of messages*

Consider the partition $\left\{ \left[0, \frac{1}{16}\right), \left[\frac{1}{16}, \frac{5}{16}\right) \left[\frac{5}{16}, 1\right] \right\}$. Suppose that the sender obeys the following strategy:

$$\begin{aligned}
\text{If } \theta &\in \left[0, \frac{1}{16}\right), \text{ randomize uniformly on } \left[0, \frac{1}{16}\right) \\
\text{If } \theta &\in \left[\frac{1}{16}, \frac{5}{16}\right), \text{ randomize uniformly on } \left[\frac{1}{16}, \frac{5}{16}\right) \\
\text{If } \theta &\in \left[\frac{5}{16}, 1\right], \text{ randomize uniformly on } \left[\frac{5}{16}, 1\right]
\end{aligned}$$

Then the receiver's best response is:

$$\begin{aligned}
\text{If } m &\in \left[0, \frac{1}{16}\right) \text{ is received, choose } a_1 = \frac{1}{8} \\
\text{If } m &\in \left[\frac{1}{16}, \frac{5}{16}\right) \text{ is received, choose } a_2 = \frac{1}{4} \\
\text{If } m &\in \left[\frac{5}{16}, 1\right] \text{ is received, choose } a_3 = \frac{5}{8}
\end{aligned}$$

Again, it is easy to check that the indifference condition for the sender boundary types is satisfied, and we have a noise equilibrium. The receiver's expected utility is $EU_R = -0.135$.

These two equilibria are illustrated in Figure 6 below. The boundary points are shown above the unit interval, and the actions chosen in each case are given below. Note that the two-step equilibrium yields substantially higher expected utility for the receiver than the three-step equilibrium.

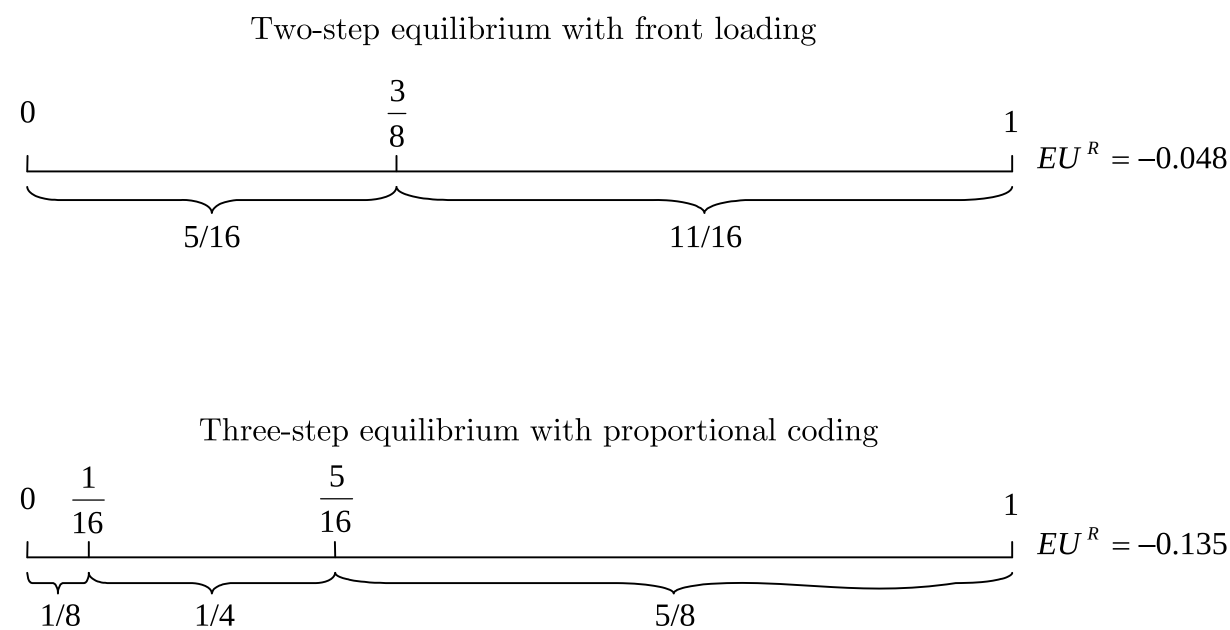


Figure 6: Equilibria with $b = \frac{1}{8}$, $\epsilon = \frac{1}{5}$

### 4.2.1 The optimal level of noise

Proposition 7 says that as long as the level of noise is low enough, we can find a noise equilibrium that Pareto dominates the best CS equilibrium. We proved this proposition by constructing a front-loading equilibrium in which the sender types in the first element of the equilibrium partition used almost all of the message space. Now suppose that we are free to choose the level of noise. Within this class of equilibria, it is easy to compute the optimal level of noise, i.e. the level which maximize the receiver's (and sender's) expected utility.

We show in the proof of Proposition 7 that the receiver's expected utility in this kind of equilibrium is given by

$$EU^R = -\frac{4b^2\left(N-2\right)\left(N-1\right)^2 N + 4b\left(N-1\right)^2\left(2N-1\right)\theta_1 + \left(\left(2N-1\right)\theta_1 - 1\right)^2}{12\left(N-1\right)^2},$$

where $N = \left\lceil \frac{1}{\sqrt{2b}} \right\rceil$ is the number of steps in the equilibrium, and $\theta_1$ is the boundary type between the first and second partition elements (see page 51 below). The value of $\theta_1$, of course, depends on the level of noise, $\epsilon$, and is given by equation 10 (page 50). Maximizing

$EU^R$ with respect to $\theta_1$ we obtain

$$\theta_1^* = \frac{1-2b(N-1)^2}{2N-1}, \text{ and hence}$$
$$\epsilon^* = \frac{\left(1-2b(N-1)^2\right)(1-2bN^2)}{4(N-1)N(b+b^2(N-1)N-1)}.$$

Figure 7 below shows $\epsilon^*$ as a function of $b$. Values of $b$ above $\frac{1}{6}$ are omitted for the sake of clarity. The function continues to rise for $b > \frac{1}{6}$, and $\epsilon^*(b) \to \frac{1}{4}$ as $b \to \frac{1}{2}$. Note that for $b = \frac{1}{2N^2}$ $(N = 2, 3, 4, \ldots)$, the optimal level of noise $\epsilon^* = 0$ — as Proposition 1 states, for these values of $b$ there is no noise equilibrium that Pareto dominates the best CS equilibrium.

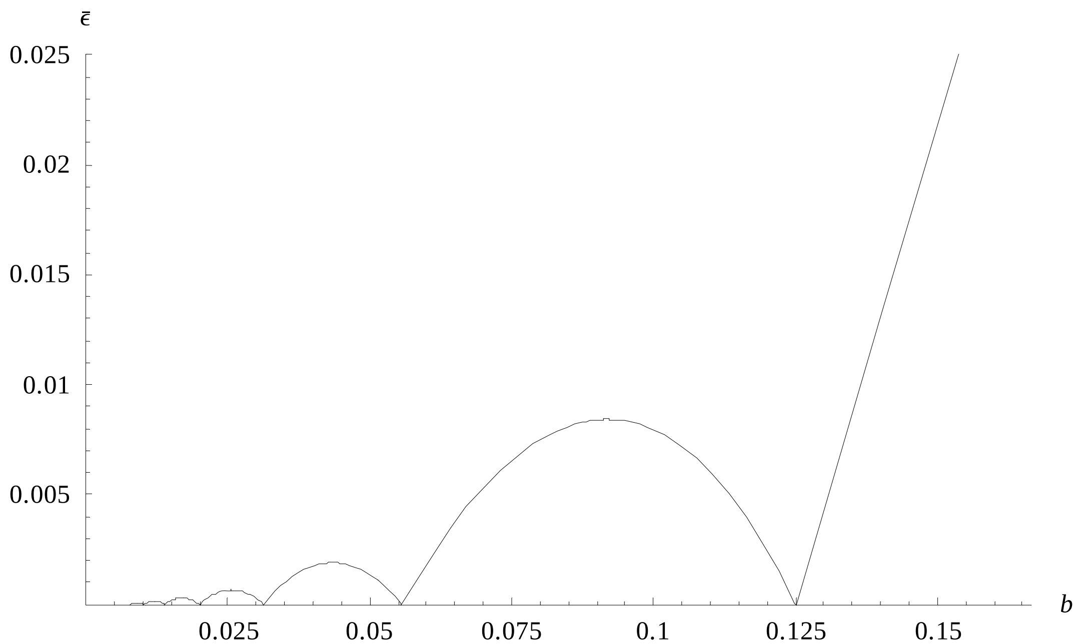


Figure 7: $\epsilon^*$ as a function of $b$

Substituting the optimal value of $\theta_1$ into the the expression for the receiver's expected utility, we get

$$EU^R = -\frac{1}{3}b(1-b).$$

This value of the expected utility is exactly the same as can be achieved by the very different equilibrium construction considered by Krishna and Morgan [14], although their construction is valid only for values of $b < \frac{1}{8}$. More significantly, Goltsman *et al.* [9] show that $-\frac{1}{3}b(1-b)$ is an upper bound on the utility that the receiver and obtain in *any* mediated equilibrium[16].

[16] That is, in any equilibrium in which the sender can submit her message to an impartial mediator, who then passes on a recommendation to the receiver according to some pre-determined and possibly stochastic

*A fortiori*, it follows that the front-loading construction with noise level $\epsilon^*$gives us the optimal noise equilibrium.

#### 4.2.2 Optimal two-step equilibria

We now return to the question of what is the optimal noise equilibrium when both $b$ and $\epsilon$ are fixed. In general, changing the coding (the way the message space is used by different sender types) changes the equilibrium partition and hence the size of the strategic effect; but it also impacts the distortion effect, since partition elements which use fewer messages induce actions which are less distorted. The interplay between these two effects is complex, so we restrict attention to the set of two-step equilibria. We believe that the optimal two-step equilibrium is front-loading, but a formal proof of this result eludes us, principally because the receiver's utility function is highly non-linear (see Figure 8 below). Intuitively, front-loading makes sense for two reasons. First, we show in section 3.1 that increasing the fraction of the message space sent by the first partition element shifts the boundary type between the two intervals to the right. As long as the boundary type is less than $\frac{1}{2}$ (which is always true for $\epsilon < \frac{4b}{1-4b}$), this makes the equilibrium partition more informative, increasing the strategic effect. Second, for a fixed partition the distortion effect is minimized when all of the message space is used by the smaller partition element; again, as long as the boundary type is less than $\frac{1}{2}$, this implies front-loading.

In Figure 8 below, we plot the receiver's equilibrium expected utility $EU^R$ against the fraction of the message space used by the first partition element, $m_1$, for different values of $b$ and $\epsilon$. Front-loading corresponds to $m_1 = 1$. Note that front-loading is optimal even when $b$ is small and $\epsilon$ is large (Figure 8d), so that the boundary type is larger than $\frac{1}{2}$.

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rule. Clearly such a mediator could reproduce the effect of noise in our model, so a noise equilibrium is a special case of a mediated equilibrium.

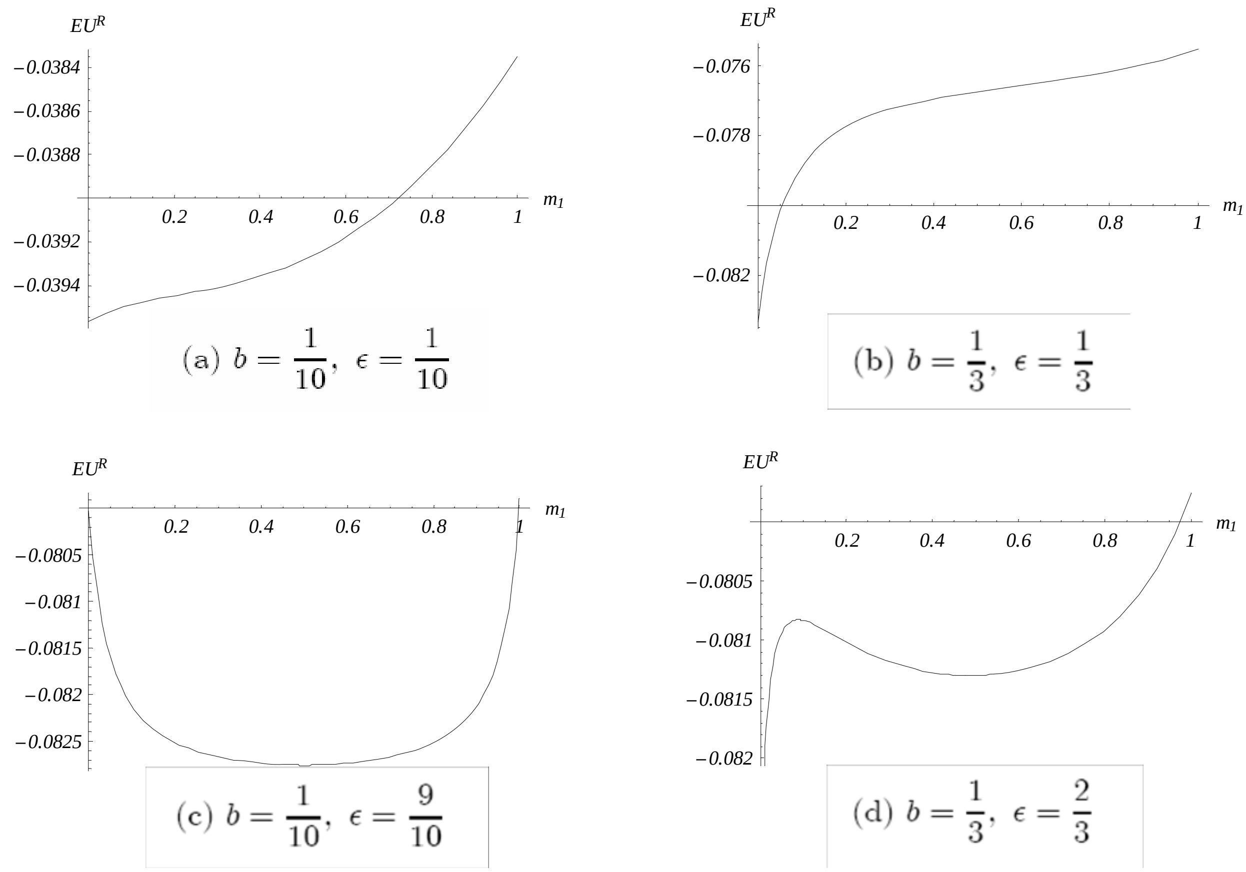


Figure 8: $EU^R$ as a function of coding

# 5 Comments and Extensions

## 5.1 Error probabilities that are correlated with messages

In this paper we have explored the impact of noise on communication, where the noise mechanism takes a very specific form: both the probability of error and what happens in the event of error are independent of the original message sent. We now consider what happens if we relax the first assumption, allowing the probability of error to vary across messages; relaxing the second assumption is left for future research.

Consider the uniform quadratic model introduced in section 3, except that the probability of error is a function of the message, $\epsilon : [0,1] \to (0,1)$, so that when the sender sends message $m$, with probability $1-\epsilon(m)$ the message is faithfully transmitted and with probability $\epsilon(m)$ the received message is a draw from the uniform distribution on $M$. We also impose a weak technical condition, assuming that $\int_0^1 \frac{1}{1-\epsilon(\lambda)} d\lambda$ is well defined. It turns out that most of the results of the preceding sections can be reproduced. Specifically, we establish the following result: suppose that there is an $N$-step front-loading noise equilibrium when noise is fixed at $\hat{\epsilon}$; then in the model with correlated noise, as long as there are $N-1$ messages which yield error probabilities close enough to $\hat{\epsilon}$, there is also an $N$-step front-loading noise equilibrium that is close to the equilibrium with fixed noise. This condition is satisfied automatically if $\epsilon(\cdot)$ is continuous and $\hat{\epsilon}$ is in its range. An immediate implication is that if $\epsilon(\cdot)$ is continuous and includes the optimal level of noise $\epsilon^*$ (derived in section 4.2.1) in its range, then we can construct an equilibrium where the sender's and receiver's expected utility are arbitrarily close to their optimal values.

Start by assuming that there is an $N$-step front-loading equilibrium in the model where noise is fixed at $\hat{\epsilon}$. Next, in the model with correlated noise, consider an $N$-step partition $\{[0,\theta_1), [\theta_1,\theta_2), \ldots, [\theta_{i-1},\theta_i), \ldots, [\theta_{N-1},1]\}$. Let $m_2, m_3, \ldots, m_N$ be a sequence of messages, with $m_i \neq m_{i'}$ for all $i \neq i'$ and $\epsilon(m_i) \geq \epsilon(m_{i+1})$ for all $i = 2, \ldots, N-1$. Define $M^* \equiv \cup_{i=2}^{N} m_i$. The sender adopts the following strategy:

$$\begin{aligned}
\text{If } \theta &\in [0,\theta_1), \text{ randomize over } M \setminus M^* \text{ with a distribution that has density } \phi \\
\text{If } \theta &\in [\theta_1,\theta_2), \text{ send message } m_2 \\
&\vdots \\
\text{If } \theta &\in [\theta_{N-1},1], \text{ send message } m_N
\end{aligned}$$

(We define $\phi$ shortly.) The receiver's posterior probability that the sender's type is in $[0,\theta_1)$ conditional on receiving a message $m \in M \setminus M^*$ equals

$$P(\theta \in [0,\theta_1) \mid m) = \frac{\left((1-\epsilon(m))\phi(m) + \int_0^1 \epsilon(\lambda)\phi(\lambda)\,d\lambda\right)\theta_1}{\left((1-\epsilon(m))\phi(m) + \int_0^1 \epsilon(\lambda)\phi(\lambda)\,d\lambda\right)\theta_1 + \sum_{i=2}^{n}\epsilon(m_i)(\theta_i - \theta_{i-1})}.$$

And the receiver's posterior probability that the sender's type is in $[\theta_{j-1},\theta_j)$ conditional on receiving a message $m \in M \setminus M^*$ equals

$$P(\theta \in [\theta_{j-1},\theta_j) \mid m) = \frac{\epsilon(m_j)(\theta_j - \theta_{j-1})}{\left((1-\epsilon(m))\phi(m) + \int_0^1 \epsilon(\lambda)\phi(\lambda)\,d\lambda\right)\theta_1 + \sum_{i=2}^{n}\epsilon(m_i)(\theta_i - \theta_{i-1})}.$$

Notice that if we can ensure that these posteriors do not vary with $m \in M \setminus M^*$, then the sender is indifferent among all messages in this set. This condition in turn is satisfied if there is a constant $c$ such that

$$(1-\epsilon(m))\phi(m) + \int_0^1 \epsilon(\lambda)\phi(\lambda)\,d\lambda = c \tag{1}$$

for all $m \in M \setminus M^*$. Integrating equation (1) with respect to $m$ shows that we must have $c = 1$. The resulting integral equation is solved by the function $\phi(\cdot)$ that is defined by

$$\phi(m) = \frac{1}{\int_0^1 \frac{1-\epsilon(m)}{1-\epsilon(\nu)} d\nu}.$$

This implies that the receiver's posteriors do not depend on the entire shape of the function $\epsilon(\cdot)$, but only on the specific values $\epsilon(m_i)$ for $i = 2, \ldots, N$. (The resulting posteriors are in fact the same as in a model where messages may get lost with probabilities $\epsilon(m_i)$, $i = 2, \ldots, N$ that depend on the messages sent, and the lowest interval of types refrains from sending a message.)

For all $m \in M \setminus M^*$ the receiver's best response is given by

$$\begin{aligned} a_1 &= P(\theta \in [0, \theta_1) \mid m) \frac{\theta_1}{2} + \sum_{i=2}^{N} P(\theta \in [\theta_{j-1}, \theta_j) \mid m) \frac{\theta_{i-1} + \theta_i}{2} \\ &= \frac{\theta_1}{\theta_1 + \sum_{i=2}^{n} \epsilon(m_i)(\theta_i - \theta_{i-1})} \frac{\theta_1}{2} + \sum_{i=2}^{N} \frac{\epsilon(m_j)(\theta_j - \theta_{j-1})}{\theta_1 + \sum_{i=2}^{n} \epsilon(m_i)(\theta_i - \theta_{i-1})} \frac{\theta_{i-1} + \theta_i}{2} \end{aligned}$$

and for $m_i \in M^*$ the receiver's best response is

$$a_i = \frac{\theta_{i-1} + \theta_i}{2}.$$

Then type $\theta$'s payoff from sending message $m_i \in M^*$ equals

$$-(1 - \epsilon(m_i))(\theta + b - a_i)^2 - \epsilon(m_i)(\theta + b - a_1)^2$$

The payoff from sending any of the messages in $M \setminus M^*$ equals

$$-(\theta + b - a_1)^2.$$

Since we have assumed $\epsilon(m_i) \geq \epsilon(m_{i+1})$, if any type $\theta$ is indifferent between two actions $a_i$ and $a_{i+1}$ with $a_{i+1} > a_i$ $(i = 1, \ldots, N-1)$, then any type $\theta' > \theta$ strictly prefers $a_{i+1}$ to $a_i$ and any type $\theta' < \theta$ strictly prefers $a_i$ to $a_{i+1}$. Therefore, if our partition is chosen such that each boundary type $\theta_i$ is indifferent between actions $a_i$ and $a_{i+1}$, we have an equilibrium. In the case where there are at least $N-1$ messages that yield equal error probabilities of exactly $\hat{\epsilon}$, the existence of such a partition is easy to show. Letting these messages be $m_2, \ldots, m_N$, so that $\epsilon(m_i) = \epsilon(m_{i'}) = \hat{\epsilon}$ for $i, i' = 2, \ldots, N$, the expression for $a_1$ simplifies to

$$a_1 = \frac{(1-\hat{\epsilon})\,\theta_1 \frac{\theta_1}{2} + \hat{\epsilon}\frac{1}{2}}{(1-\hat{\epsilon})\,\theta_1 + \hat{\epsilon}},$$

which is exactly the same as in the model where noise is fixed at $\hat{\epsilon}$. It follows that the indifference conditions for each boundary type are the same, and hence the equilibrium partition of the fixed noise model is also an equilibrium partition of the correlated noise model. In the case where we cannot find $N-1$ messages that yield equal error probabilities of $\hat{\epsilon}$ (for instance if $\epsilon(\cdot)$ is strictly increasing), such an equilibrium exists as long as we can find $N-1$ messages that yield error probabilities close enough to $\hat{\epsilon}$. If $\epsilon(\cdot)$ is continuous and includes $\hat{\epsilon}$ in its range, then we can find $N-1$ messages that yield error probabilities arbitrarily close to $\hat{\epsilon}$ and the existence results follows from a simple continuity argument.

As an example, let $b = \frac{1}{10}$ (so there are at 2 steps in any CS equilibrium, but there is a front-loading (fixed) noise equilibrium with 3 steps when $\epsilon = 0.08$) and consider any error function $\epsilon(\cdot)$ for which there are two messages $m_2$ and $m_3$ such that $\epsilon(m_2) = 0.08$ and $\epsilon(m_3) = 0.07$. Then there is a three-step noise equilibrium with boundary values $\theta_1 = 0.137$ and $\theta_2 = 0.369$, equilibrium, and actions $a_1 = 0.222$, $a_2 = 0.253$ and $a_3 = 0.684$ induced by the first, second and third partition elements respectively.

## 5.2 Sender error

The model we have analyzed in this paper assumes that there is a possibility of error in the information transmission process, so that the message received may differ from the message sent. But it is also natural to investigate what happens if the error occurs at an earlier stage, before the sender receives her information. In this section we consider a model in which, with probability $1-\eta$ the sender observes a signal which is equal to her type, and with probability $\eta$ she observes a random signal that is a draw from the uniform distribution on the type space, $[0,1]$. We assume that $0 < \eta < 1$, and that the sender is unable to distinguish the accurate signal from the random signal.

Equilibria in this *sender-error model* are very similar to equilibria in the CS model, with the type space divided into a finite number of intervals and the sender's message revealing only in which interval her signal lies. We show here that the best CS equilibrium Pareto dominates every sender-error equilibrium: unlike noisy messages, noisy signals for the sender can only worsen the prospects for information transmission.

Consider a partition of the type space into $N$ intervals, $\{[0,\theta_1), [\theta_1,\theta_2), \ldots, [\theta_{N-1},1]\}$, and suppose that types in the $i$th interval randomize uniformly over messages in $[\theta_{i-1},\theta_i)$. (Note that in the sender-error model, as in the CS model, the set of messages and the distribution over this set is not important, as long as a distinct set of messages is used by each element of the type space partition.) The receiver's best response, $a_i$, to a message $m \in [\theta_{i-1}, \theta_i)$ is a weighted average of his expectation of $\theta$ given that the sender's signal was accurate and his expectation of $\theta$ given that the signal was an error:

$$a_i = (1-\eta)\frac{\theta_{i-1}+\theta_i}{2} + \eta\frac{1}{2}.$$

The sender's expected utility if she sends a message $m \in [\theta_{i-1}, \theta_i)$ when her signal is $\theta$ is given by

$$E\left[U^S(\theta, a_i, b)\right] = -(1-\eta)(\theta+b-a_i)^2 - \eta\int_0^1 (\theta'+b-a_i)^2\, d\theta'.$$

As usual, equilibrium requires that the sender boundary types $\theta_i$ are indifferent between inducing action $a_{i-1}$ and action $a_i$:

$$\begin{aligned} U^S(\theta_i, a_{i-1}, b) &= U^S(\theta_i, a_i, b) \\ \Rightarrow \theta_{i+1}-\theta_i &= \theta_i - \theta_{i-1} + \frac{4b}{1-\eta}. \end{aligned} \tag{2}$$

Equation 2 tells us that each interval must be $\frac{4b}{1-\eta}$ longer than the previous interval. Since in the CS model the intervals grow by only $4b$ each time, the most informative equilibrium partition in the sender-error model must be less-evenly spaced and contain weakly fewer elements. Combining this with the direct loss of information when the sender gets a false signal results in a utility loss for both agents. The largest number of steps in any equilibrium of the CS model and the sender error model are given by

$$N_{CS}(b) = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{2}{b}} \right\rceil \quad \text{and} \quad N_{SE}(b) = \left\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{2(1-\eta)}{b}} \right\rceil,$$

respectively (where $\lceil x \rceil$ is the smallest integer than is greater than or equal to $x$); and the receiver's expected utilities in the Pareto optimal equilibrium in each case are

$$\begin{aligned} EU^R_{CS} &= -\frac{1}{12N_{CS}(b)^2} - \frac{b^2\left(N_{CS}(b)^2-1\right)}{3} \quad \text{and} \\ EU^R_{SE} &= -\frac{(1-\eta)^2}{12N_{SE}(b)^2} - \frac{b^2\left(N_{SE}(b)^2-1\right)}{3\eta^2} - \frac{(2-\eta)\eta}{12}. \end{aligned}$$

It is easy to show that $EU^R_{CS} > EU^R_{SE}$ for $0<\eta<1$ and $0<b<\frac{1}{4}$, and also that $EU^R_{SE}$ is strictly decreasing in $\eta$.

The sender-error model shows that reducing the accuracy of the sender's signal stifles communication with the receiver, resulting in a loss of utility for both. An alternative analysis of inaccurate signals is provided by Fischer and Stocken [6] and Ivanov [11], who obtain the opposite result with a different assumption about the sender's signal. Specifially, they assume that the type space is partitioned into a number of intervals, and the sender's signal reveals only in which element her actual type lies (for example, the signal might indicate whether her type lies in the first, second, third or fourth quarter of the unit interval). Ivanov shows that if the sender's information can be restricted in this way, and the information structure is chosen optimally, then there are equilibria which Pareto dominate the best CS equilibrium if and only if $b \leq \frac{1}{4}$.

## 5.3 Noisy talk with common interest

Although the focus of this paper is on the interaction between noise and divergent interests, it is instructive to consider the effects of noise in the common interest case, where $b = 0$. Recall that an equilibrium is separating if every sender type induces a different outcome. Without noise, there is a separating equilibrium where the sender follows the "natural" strategy of sending message $m = \theta$ when her type is $\theta$, and the receiver chooses action $a = m$. But this is not an equilibrium when there is noise. The reason is that under this sender strategy, the posterior probability of an error having occurred equals the prior, $\epsilon$. Thus, following each message the receiver will attribute this probability to the event that this message was the result of noise and therefore distort his response toward the pooling response, choosing action $a = (1 - \epsilon) m + \epsilon \frac{1}{2}$. A rational sender would try to offset this distortion by deviating from the rule $m = \theta$. This illustrates nicely the distortionary effect of introducing noise: Even in the common-interest game sender and receiver cannot simply continue to use the strategies that "work" in the absence of noise. Our next result shows that nevertheless a separating equilibrium does exist for any value of the error probability $\epsilon$. The fundamental idea underlying the construction of such an equilibrium is to have the sender use only a small subset of the message space.[17] Then whenever a message from this subset is received, the posterior probability that it was not sent by error is high. As the size of the set of used messages converges to zero, this posterior probability converges to one. Denote the common expected payoff in a separating equilibrium of the common-interest game with error probability $\epsilon$ by $\Pi(\epsilon)$. Note that the pooling payoff, $\Pi_p$, is independent of the error probability (in the uniform quadratic case considered here, $\Pi(0) = 0$ and $\Pi_p = -\frac{1}{12}$, but it is easy to see that the proof is easily extended to the more general model introduced in

[17]Jakub Steiner has pointed out to us that credit card companies use the same technique to reduce the likelihood that one card number is mistaken for another: Actual credit card numbers are a small subset of the set of possible numbers with a fixed number (usually 16) of digits. This makes it possible to identify mistaken entries with high probability. Often, the final digit is determined from the previous digits by means of the Luhn-10 algorithm, which detects almost all transpositions of adjacent numbers.

section 2.1).

**Proposition 9** *The common interest game has a separating equilibrium. In this equilibrium the receiver learns with probability one whether or not a received message was sent in error. Equilibria of this form are efficient in the common interest game and have a common expected payoff*

$$\Pi(\epsilon) = (1-\epsilon)\,\Pi(0) + \epsilon\Pi_p.$$

**Proof.** Take any set $M^0 \subset [0,1]$ that has the same cardinality as the set $[0,1]$ and at the same time has (Lebesgue) measure zero (for example, $M^0$ can be the Cantor set). Since $M^0$ has the same cardinality as $[0,1]$, there exists a sender strategy that is a bijection from the type space $[0,1]$ in to $M^0$. At the same time, since the the error distribution has a density, sets of (Lebesgue) measure zero have probability zero; it follows from Bayes' rule that the probability of an error following a message $m \in M^0$ equals zero. Therefore, whenever he receives a message in $M^0$ the receiver knows that with probability one the message was not sent in error. In that case, given that the sender's strategy is a bijection, the receiver correctly infers the sender's type. Similarly, whenever he receives a message in $[0,1]\setminus M^0$, the receiver knows that with probability one the message was sent in error. Regarding efficiency, it suffices to consider receiver payoffs. Conditional on each event, noise or no-noise, the receiver maximizes his payoff. Therefore he maximizes his *ex ante* expected payoff. The fact that the expected payoff $\Pi(\epsilon)$ has the indicated form is a simple consequence of the receiver taking the separating action in the no-noise event, and the pooling action in the noise event. Finally, note that since all sender types receive their ideal action in the no-noise event and cannot affect the action taken in the noise event, the sender has no incentive to deviate. ■

As an aside, we observe that Theorem 4 of Gordon [10] can be used to show that there is an alternative, *proportional-coding* equilibrium (see example 2 in section 4.2 above) with an infinite partition, in which types in partition element $[\theta_{j-1},\theta_j)$ randomize uniformly over the interval $[\theta_{j-1},\theta_j)$. These equilibria have the the intuitive property that types in a given element of the equilibrium partition use only messages in the same set. Since these equilibria do not make optimal use of the available information, however, unlike equilibria in which the set of used messages has measure zero, they are not Pareto optimal. Therefore, we have the additional observation that in the common-interest game there are multiple Pareto-ranked infinite-interval partition equilibria.

Returning to the construction used to prove Proposition 9, a referee has observed that it relies on the message space being a continuum, so we can find an uncountable set of messages that nevertheless has measure zero. In the finite case, separating equilibria may not exist if the cardinality of the message space is close to the cardinality of the type space and the level of noise is sufficiently high. But we now show that, as long as the message space is large enough, separation can be achieved through a construction analogous to that used in the infinite case.

Let $K$ be a positive integer and consider the finite set of types $T(K) = \left\{\theta \in [0,1] \mid \theta = n \times \frac{1}{K} \text{ for some } n \in \mathbb{N}_0\right\}$, each of which is equally likely. First, to show why separating may be impossible, let the message space be identical to the type space[18], and suppose the error distribution is uniform. In a candidate separating equilibrium, the receiver's response to the message sent by the lowest type ($\theta = 0$) would be the action $a_0 = (1-\epsilon)\,0 + \epsilon\frac{1}{2}$, while his best response to the message sent by the second lowest type would be $a_1 = (1-\epsilon)\frac{1}{K} + \epsilon\frac{1}{2}$. For incentive compatibility, type $\theta = \frac{1}{K}$ must prefer $a_1$ to $a_0$, i.e.

$$\begin{aligned} -\left(\frac{1}{K} - a_1\right)^2 &\geq -\left(\frac{1}{K} - a_0\right)^2 \\ \Rightarrow \epsilon &\leq \frac{1}{K-1}. \end{aligned}$$

If $\epsilon$ is above this threshold, then, no separating equilibrium exists (in fact, this condition is a necessary and sufficient condition for the existence of a separating equilibrium). By way of an example, suppose $K = 4$ and $\epsilon = \frac{1}{2} > \frac{1}{K-1}$. In a separating equilibrium the receiver's response to the messages sent by the lowest two types $\theta = 0$ and $\frac{1}{4}$ would be the actions $\frac{1}{4}$ and $\frac{3}{8}$ respectively; clearly type $\theta = \frac{1}{4}$ would have an incentive to deviate and mimic type $\theta = 0$.

If we fix $K$ and $\epsilon$, however, but increase the size of the message space, we can always find a message space large enough that a separating equilibrium exists. Suppose that each sender type sends exactly one (distinct) message. Then as the message space grows, the receiver's response to a message sent by type $\theta$ converges to $\theta$ for all $\theta \in T(K)$. Formally, if the receiver observes the message sent by type $\theta$, his best response is given by

$$a_\theta = (1-\mu)\,\theta + \mu\frac{1}{2},$$

where

$$\mu = \frac{\epsilon\frac{1}{|M|}}{(1-\epsilon)\frac{1}{K+1} + \epsilon\frac{1}{|M|}}$$

is the receiver's posterior probability that the message was sent in error. As $|M| \to \infty$, $\mu \to 0$ and so $a_\theta \to \theta$. Hence, for a sufficiently large message space, each sender type strictly prefers to send the message assigned to her then any message sent by the other types, or one of the unsent messages (to which the receiver's best response is action $a = \frac{1}{2}$).

[18] If the message space is smaller than the type space, separation is impossible for a trivial reason.

# 6 Conclusion

In this paper we have examined two principal barriers to communication, misaligned preferences and the possibility of misunderstandings, and their interaction. We find that while each of these factors limits communication on its own, the possibility of misunderstandings may help partially overcome the limitations due to divergent preferences. We have shown that introducing a small amount of noise into information transmission can almost always benefit communication. When noise levels continuously vary across a sufficiently large range of messages, there are equilibria that approximate optimal mediated communication. In the case of extreme biases, introducing noise may enable communication when it would otherwise not have been possible.

In natural language there is a variety of sources of noise, e.g. the need to interpret messages, politeness conventions, and vagueness. We believe that our result, that small amounts of noise in communication can almost always improve efficiency, suggests a partial answer to the question posed by Lipman [15]: "Why is language vague?"[19]

A word is vague when there are borderline cases where it is not clear whether or not it applies. Such words are ubiquitous in natural language: examples include "tall", "red", "child" and "heap". We believe that the noise model helps us understand one important reason for the frequent use of vague words in natural language: Access to vague words, and vague language use more generally, facilitates incentive compatible communication when there is conflict of interest.

By way of an example, suppose that Romeo has asked Juliet whether she loves him. He knows that Juliet can be of two types, one who passionately loves Romeo, and one who is merely fond of him as a friend. Juliet-the-lover wants to communicate that fact to him, but would like him to retain enough uncertainty to keep him keen. Juliet-the-friend simply wishes to remain on good terms, but knows that he will never talk to her again if he finds out that she definitely doesn't love him.

If we model this scenario as a conventional sender-receiver game, it is easily seen that there is no pure strategy for Juliet that would permit her to induce the desired beliefs in both cases. Any pure strategy either identifies both Juliet-the-lover and Juliet-the-friend, or reveals nothing. Juliet can do better by using a mixed strategy. If she randomizes between "I love you" and "I like you" when she loves Romeo and announces "I like you" otherwise, Romeo will be pleased by "I love you" and may remain hopeful after "I like you." Of course, mixing in this fashion is not incentive compatible for Juliet-the-lover.

This is where vagueness can help. In a given context, the meaning, and hence the correct

[19] A similar answer can be found in De Jaegher [5], who supports his reasoning with an example based on the coordinated attack problem, rather than in the context of sender-receiver games.

interpretation, of a vague word may depend on the language habits of the utterer.[20] Here, when Juliet says "I love you", Romeo understands that she is probably in love with him, but also that these words are occasionally used to express mere fondness. On the other hand, when Juliet say "I like you", Romeo realizes that she is probably just a friend, but retains a slim hope that he is facing a lover who chooses her words conservatively.

To link this story back to the noise model, we think of the words uttered by Juliet as corresponding to the message sent, while Romeo's interpretation of those words corresponds to the message received. In essence, we operationalize one aspect of vague language use, the interpretability of vague messages, by identifying it with communicating through a noisy channel.

This is in a similar spirit to Lipman [15], who entertains the use of mixed strategies in common-interest sender-receiver games as a possible expression of vagueness. He links mixing to vagueness because "With such randomization, player 2's interpretation of a given message would be that 1 is more likely to use this message to describe some [types] than others." Lipman shows that mixing in common-interest games can never benefit the agents and therefore cannot provide a rationale for vagueness. We depart from Lipman in two directions, by permitting communication through noisy channels rather than just mixing, and by dropping the common-interest condition.

One drawback of focusing on plain mixed strategies, we believe, is that in equilibrium at the moment the sender sends a message she knows the exact interpretation of the message. In contrast, when uttering a vague word like "tall" the speaker cannot be sure how it will be interpreted by the listener. This feature is captured by our noise model, if we identify the sent message with the speaker's utterance and the received message with the listener's interpretation of the utterance. In a noise equilibrium, while there is a most likely interpretation of a message, no interpretation can be ruled out.

We believe that it is useful to include situations in which interests are not fully aligned in the study of vagueness. Economists sometimes connect vagueness with contractual incompleteness. Contracting, however, has both common-interest and adversarial components, since it affects the division of surplus.

Our formalization of vagueness, one that identifies it with communication via a noisy channel, establishes a role for vagueness in any neighborhood of the class of common interest games: For (almost) any positive bias that is not too large, there is a vague language that Pareto dominates a precise language. Moreover, when error probabilities are correlated with

[20] See e.g. Peirce [21]: "A proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker's habits of language were indeterminate; so that one day he would regard the proposition as excluding, another as admitting, those states of things."

messages, so that both noisy and noiseless message are available, the best equilibrium with a precise language is dominated by an equilibrium with a vague language.

Hints of this rationale for vagueness can be found in the management literature. For example, in their study of Japanese management, Pascale and Athos [20] write: "*Vagueness in communication* can cause problems, to be sure, but it can also serve to hold strained relations together and reduce unnecessary conflict."

# A Proofs and calculations

Before proving Proposition 1, we start with two lemmas:

**Lemma 1** *In every noise equilibrium, for every action $a$, the set of sender types who induce action $a$ is an interval. If this interval has a nonempty interior, then all types in the interior induce only action $a$.*

**Proof.** If $a$ is induced by only one type, the result holds trivially. If types $\theta$ and $\theta'$ with $\theta < \theta'$ both induce action $a$, then types $\theta'' \in (\theta, \theta')$ never induce an action $a_1 > a$, because otherwise, since $U_{12}^S(a, \theta, b) > 0$, type $\theta'$ would strictly prefer $a_1$ to $a$; similarly, types $\theta'' \in (\theta, \theta')$ never induce an action $a_0 < a$, because otherwise by single crossing type $\theta$ would strictly prefer $a_0$ to $a$. Hence types in the interval $(\theta, \theta')$ induce only action $a$. ■

**Lemma 2** *If type $\theta$ induces actions $a_1$ and $a_2$ with $a_1 < a_2$, then there exists $\eta > 0$ such that types in $(\theta - \eta, \theta)$ induce action $a_1$ and types in $(\theta, \theta + \eta)$ induce action $a_2$.*

**Proof.** Concavity of the sender's payoff function in $a$ implies that $a_1 < a^S(\theta, b) < a_2$ (where $a^S(\theta, b)$ denote's type $\theta$'s ideal action). Continuity of the sender's payoff function and the single crossing condition ($U_{12}^S > 0$) imply that there is a nonempty open set of types $(\theta, \theta + \eta_1)$ such that for all $\theta' \in (\theta, \theta + \eta_1)$ we have $a^S(\theta, b) < a^S(\theta', b) < a_2$. Type $\theta'$'s payoff is decreasing to the right of $a_2$ and by single crossing, $\theta'$ prefers $a_2$ to all actions $a \in (-\infty, a_1)$. No action $a \in (a_1, a_2)$ is induced in equilibrium because otherwise type $\theta$ would have an incentive to deviate. This shows that all types in $(\theta, \theta + \eta_1)$ must induce action $a_2$. An analogous argument shows that we can find a nonempty open set $(\theta - \eta_2, \theta)$ such all types in that set induce action $a_1$. Choose $\eta = \min\{\eta_1, \eta_2\}$. ■

**Proof of Proposition 1.** The main result now follows easily: Lemma 2 implies immediately that there is at most a countable number of types who induce two actions and that the receiver's response is not altered if we have all such types switch to induce only one these actions. ■

**Proof of Proposition 2.** The receiver's payoff from choosing action $a$ conditional on observing message $m \in M_i$ is given by

$$\int_0^1 U^R(a, \theta)\mu(\theta|m)d\theta.$$

Maximizing this expression with respect to $a$ is equivalent to maximizing

$$\int_0^1 U^R(a, \theta)((1 - \epsilon)\sigma(m|\theta) + \epsilon g(m))f(\theta)d\theta.$$

Since $a(m)$ is a common maximizer for all $m \in M_i$, it also maximizes

$$\begin{aligned} & \int_{M_i} \int_0^1 U^R(a,\theta)((1-\epsilon)\sigma(m|\theta) + \epsilon g(m)) f(\theta) d\theta dm \\ = & \int_{\theta_{i-1}}^{\theta_i} U^R(a,\theta)(1-\epsilon) f(\theta) d\theta + \int_0^1 U^R(a,\theta)\epsilon \int_{M_i} g(m) dm f(\theta) d\theta \end{aligned}$$

Maximizing the latter expression, however, is equivalent to maximizing

$$\int_{\theta_{i-1}}^{\theta_i} U^R(a,\theta)(1-\epsilon)\frac{g(m)}{\int_{M_i} g(m)dm} f(\theta)d\theta + \int_0^1 U^R(a,\theta)\epsilon g(m) f(\theta) d\theta,$$

which is exactly the problem that the receiver solves after receiving a message $m \in M_i$ when all types in $(\theta_{i-1}, \theta_i)$ use the common distribution $\frac{g(m)}{\int_{M_i} g(m)dm}$ on $M_i$. ■

**Proof of Proposition 3.** First, assume that there is a type $\theta_0$ whose ideal action is the pooling action, $a_p$. Suppose there is a set $M_0$ of unused messages that has positive measure. Whenever the receiver observes a message $m_0 \in M_0$, the receiver's optimal reply $a(m_0)$ satisfies

$$a(m_0) = \arg\max_a \int_0^1 U^R(a,\theta) f(\theta) d\theta.$$

By assumption $a(m_0)$ is the ideal action for type $\theta_0$. Since this type could induce this action by sending one of the unused messages, he induces it in equilibrium. Consider first any equilibrium in which the set of types, $\Theta_0$, who induce the same action $a(m_0)$ as $\theta_0$ has positive probability. Using $M(\Theta_0)$ to denote the set of messages used by $\Theta_0$, we have

$$a(m_0) = \arg\max_a \int_{\Theta_0} U^R(a,\theta)(1-\epsilon)\frac{g(m)}{\int_{M(\Theta_0)} g(m)dm} f(\theta)d\theta + \int_0^1 U^R(a,\theta)\epsilon g(m) f(\theta) d\theta.$$

Since $a(m_0)$ maximizes the second term of this expression, it must also maximize the first term. Therefore, we also have

$$a(m_0) = \arg\max_a \int_{\Theta_0} U^R(a,\theta)(1-\epsilon)\frac{g(m)}{\int_{M_0 \cup M(\Theta_0)} g(m)dm} f(\theta)d\theta + \int_0^1 U^R(a,\theta)\epsilon g(m) f(\theta) d\theta.$$

Next consider any equilibrium in which the set of types $\Theta_0$ has probability zero. Having all these types randomize uniformly over $M_0 \cup M(\Theta_0)$ does not alter the receiver's posterior after any message and thus preserves equilibrium.

Now we deal with the case where there is no type whose ideal action is the pooling action. Since $b > 0$ and $U^S(\cdot,\cdot,\cdot)$ is continuous, every type's ideal action must be larger than $a_p$ ($a^S(\theta, b) > a_p$ for all $\theta \in T$). If the equilibrium under consideration is pooling, the conclusion follows immediately. If not, then we can find two actions, $a_1$ and $a_2$, induced in equilibrium with the property that $a_1 < a_p < a_2$. It follows easily that there cannot be any unused messages: sending an unused message would induce action $a_p$, which is preferred by every type to action $a_1$. ■

**Proof of Proposition 4.** First note that the result is immediate if $\mathcal{M}$ and $\mathcal{M}'$ have a different number of elements, so we assume they both have $N$ elements. Consider a noise equilibrium that is adapted to $\mathcal{M}$, where $T_1, \ldots, T_N$ is the equilibrium partition. Let $\theta_{i-1}$ and $\theta_i$ be the lower and upper bound of $T_i$, and let $m_{i-1}$ and $m_i$ be the lower and upper bound of $M_i$ (the $i$th element of $\mathcal{M}$). We use $\lambda(M_i) = G(m_i) - G(m_{i-1})$ to denote the probability that a message from $M_i$ is received in the error event. Define

$$a(\theta_{i-1}, \theta_i, M_i) := \arg\max_a \int_{\theta_{i-1}}^{\theta_i} U^R(a,\theta)(1-\epsilon) f(\theta)\, d\theta + \int_0^1 U^R(a,\theta)\, \epsilon\lambda(M_i) f(\theta) d\theta.$$

Since types in $T_i$ randomize with the error distribution over messages in $M_i$, the receiver's equilibrium response $a_i$ to a message $m \in M_i$ satisfies $a_i = a(\theta_{i-1}, \theta_i, M_i)$. Furthermore, the arbitrage conditions

$$U^S(a_i, \theta_i, b) = U^S(a_{i+1}, \theta_i, b), \quad i = 0, \ldots, n-1$$

must hold for the sender. Hence, the boundary types $\theta_1, \ldots, \theta_{N-1}$ must satisfy the second-order difference equation

$$U^S(a(\theta_{i-1}, \theta_i, M_i), \theta_i, b) = U^S(a(\theta_i, \theta_{i+1}, M_{i+1}), \theta_i, b).$$

Notice that for the given equilibrium there is at most one interval $T_i$ for which

$$\arg\max_a \int_{\theta_{i-1}}^{\theta_i} U^R(a,\theta) f(\theta)\, d\theta = \arg\max_a \int_0^1 U^R(a,\theta) f(\theta)\, d\theta,$$

and that for any $\mathcal{M}'$ that is distinct from $\mathcal{M}$, there must be at least two corresponding pairs of intervals, $i$ and $j$, such that $\lambda(M_i) \neq \lambda(M_i')$ and $\lambda(M_j) \neq \lambda(M_j')$. It follows immediately that $O(\mathcal{M}) \cap O(\mathcal{M}') = \emptyset$, as required. ■

Before proving Proposition 5, we provide a formal restatement of Crawford and Sobel's monotonicity condition (M). In the CS model, let $a_{CS}(\theta_{i-1},\theta_i)$ denote the receiver's best response to a message that indicates only that the sender's type lies in $(\theta_{i-1},\theta_i)$, i.e.

$$a_{CS}(\theta_{i-1},\theta_i) = \arg\max_{a'} \int_{\theta_{i-1}}^{\theta_i} U^R(a,\theta) f(\theta)\, d\theta.$$

Consider the second-order difference equation

$$U^S\left(a_{CS}\left(\theta_{i-1},\theta_i\right),\theta_i,b\right)=U^S\left(a_{CS}\left(\theta_i,\theta_{i+1}\right),\theta_i,b\right). \tag{3}$$

Then condition (M) says:

**(M)** Suppose $(\theta_0,\theta_1,\ldots,\theta_N)$ and $(\theta_0',\theta_1',\ldots,\theta_N')$ are two solutions to equation 3, and $\theta_0=\theta_0'$ and $\theta_1>\theta_1'$. Then $\theta_i>\theta_i'$ for all $i\geq 2$.

**Proof of Proposition 5.** Denote the boundary types of the $N$-step CS partition by $\theta_i^*$, $i=1,\ldots,N-1$. Let $\hat{a}(\theta_{i-1},\theta_i,\mathcal{M},\epsilon)=a(\theta_{i-1},\theta_i,M_i)$ , where $M_i$ is the $i$th component of $\mathcal{M}$ and $a(\theta_{i-1},\theta_i,M_i)$ is as defined in the proof of Proposition 4 above (so $\hat{a}(\theta_{i-1},\theta_i,\mathcal{M},\epsilon)$ is the receiver's best response to messages in $M_i$ when types in $(\theta_{i-1},\theta_i)$ randomize over messages in that set according to the error distribution and when $\epsilon$ is the level of noise). We can view $\mathcal{M}$ as a point in the $N-1$ simplex $\triangle^{N-1}$. Then, using the notation $\langle x_1,\ldots,x_n\rangle$ to denote an ordered $n$-tuple, $\hat{a}\left(\cdot,\cdot,\cdot,\cdot\right)$ is a continuous function on the set $\{\langle\theta_{i-1},\theta_i\rangle\mid\theta_{i-1}\leq\theta_i\}\times\Delta^{N-1}\times[0,1]$; this follows from continuity of $U^R\left(\cdot,\cdot\right)$, the theorem of the maximum and uniqueness of $\hat{a}\left(\cdot,\cdot,\cdot,\cdot\right)$.

Next, define

$$V\left(\theta_{i-1},\theta_i,\theta_{i+1},\mathcal{M},\epsilon,b\right)\equiv U^S\left(\hat{a}\left(\theta_i,\theta_{i+1},\mathcal{M},\epsilon\right),\theta_i,b\right)-U^S\left(\hat{a}\left(\theta_{i-1},\theta_i,\mathcal{M},\epsilon\right),\theta_i,b\right).$$

Continuity of $\hat{a}\left(\cdot,\cdot,\cdot,\cdot\right)$ and of $U^S\left(\cdot,\cdot,\cdot\right)$ implies that $V\left(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot\right)$ is continuous on the compact set $\{\langle\theta_{i-1},\theta_i,\theta_{i+1}\rangle\mid\theta_{i-1}\leq\theta_i\leq\theta_{i+1}\}\times\triangle^{N-1}\times[0,1]\times\{b\}$ (whatever the value of $b$). Therefore $V\left(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot\right)$ is uniformly continuous on this set.

$\hat{a}\left(\theta_i,\theta_{i+1},\mathcal{M},\epsilon\right)$ is strictly increasing in its first two arguments; this is a consequence of the single-crossing condition $U_{12}^R>0$.

$V\left(\theta_{i-1}^*,\theta_i^*,\tau_{i+1},\mathcal{M},0,b\right)$ is a strictly decreasing function of $\tau_{i+1}$ in a neighborhood of $\theta_{i+1}^*$. This follows from $V\left(\theta_{i-1}^*,\theta_i^*,\theta_i^*,\mathcal{M},0,b\right)>0$, $V\left(\theta_{i-1}^*,\theta_i^*,\theta_{i+1}^*,\mathcal{M},0,b\right)=0$, $\hat{a}\left(\theta_i,\theta_{i+1},\mathcal{M},0\right)$ strictly increasing in $\theta_{i+1}$ and strict concavity of $U^R\left(a,\theta\right)$ in $a$. This implies that there exist $\tau_{i+1}'$ and $\tau_{i+1}''$ with $\tau_{i+1}'<\theta_{i+1}^*<\tau_{i+1}''$ such that $V\left(\theta_{i-1}^*,\theta_i^*,\tau_{i+1}',\mathcal{M},0,b\right)>0>V\left(\theta_{i-1}^*,\theta_i^*,\tau_{i+1}'',\mathcal{M},0,b\right)$. This and the uniform continuity of $V\left(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot\right)$ on $\{\langle\theta_{i-1},\theta_i,\theta_{i+1}\rangle\mid\theta_{i-1}\leq\theta_i\leq\theta_{i+1}\}\times\triangle^{N-1}\times[0,1]\times\{b\}$ implies that there exists $\eta_1>0$ such that for all $\tau_{i-1}\in\left[\theta_{i-1}^*-\eta_1,\theta_{i-1}^*+\eta_1\right]$, for all $\tau_i\in\left[\theta_i^*-\eta_1,\theta_i^*+\eta_1\right]$, for all $\epsilon\in[0,\eta_1]$ and for all $\mathcal{M}$, we have $V\left(\tau_{i-1},\tau_i,\tau_{i+1}',\mathcal{M},\epsilon,b\right)>0>V\left(\tau_{i-1},\tau_i,\tau_{i+1}'',\mathcal{M},\epsilon,b\right)$. Hence, the intermediate value theorem implies that for all $\tau_{i-1}\in\left[\theta_{i-1}^*-\eta_1,\theta_{i-1}^*+\eta_1\right]$, for all $\tau_i\in\left[\theta_i^*-\eta_1,\theta_i^*+\eta_1\right]$, for all $\epsilon\in[0,\eta_1]$ and for all $\mathcal{M}$, there exists $\tau_{i+1}\left(\tau_{i-1},\tau_i,\mathcal{M},\epsilon\right)$ such that $V\left(\tau_{i-1},\tau_i,\tau_{i+1}\left(\tau_{i-1},\tau_i,\mathcal{M},\epsilon\right),\mathcal{M},\epsilon,b\right)=0$.

Furthermore, from $V\left(\tau_{i-1},\tau_i,\tau'_{i+1},\mathcal{M},\epsilon,b\right) > 0 > V\left(\tau_{i-1},\tau_i,\tau''_{i+1},\mathcal{M},\epsilon,b\right)$, the fact that $\hat{a}\left(\tau_i,\tau_{i+1},\mathcal{M},\epsilon\right)$ is strictly increasing in $\tau_{i+1}$ and strict concavity of $U^R\left(a,\theta\right)$ in its first argument, it follows that $\tau_{i+1}\left(\tau_{i-1},\tau_i,\mathcal{M},\epsilon\right)$ is unique. In conjunction with the continuity of $V\left(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot\right)$, this implies that $\tau_{i+1}\left(\tau_{i-1},\tau_i,\mathcal{M},\epsilon\right)$ is continuous for all $\tau_{i-1}\in\left[\theta^*_{i-1}-\eta_1,\theta^*_{i-1}+\eta_1\right]$, for all $\tau_i\in\left[\theta^*_i-\eta_1,\theta^*_i+\eta_1\right]$, for all $\epsilon\in[0,\eta_1]$ and for all $\mathcal{M}$.

Iterating on $i$, this implies that there exists $\eta>0$ such that for all $\mathcal{M}$, for all $\theta_1$ with $|\theta_1-\theta^*_1|\leq\eta$ and $\epsilon$ that satisfy $\epsilon\leq\eta$, there exists a solution $\theta_i\left(\theta_1,\mathcal{M},\epsilon\right)$ to the difference equation

$$U^S\left(\hat{a}\left(\theta_{i-1},\theta_i,\mathcal{M},\epsilon\right),\theta_i,b\right)=U^S\left(\hat{a}\left(\theta_i,\theta_{i+1},\mathcal{M},\epsilon\right),\theta_i,b\right)$$

with initial values $\theta_0=0$ and $\theta_1$, and that the solution is continuous on this domain.

Define

$$\begin{aligned}W\left(\theta_1,\mathcal{M},\epsilon\right) \equiv\; & U^S\left(\hat{a}\left(\theta_{n-1}\left(\theta_1,\mathcal{M},\epsilon\right),1,\mathcal{M},\epsilon\right),\theta_{n-1}\left(\theta_1,\mathcal{M},\epsilon\right),b\right)\\ & -U^S\left(\hat{a}\left(\theta_{n-2}\left(\theta_1,\mathcal{M},\epsilon\right),\theta_{n-1}\left(\theta_1,\mathcal{M},\epsilon\right),\mathcal{M},\epsilon\right),\theta_{n-1}\left(\theta_1,\mathcal{M},\epsilon\right),b\right)\end{aligned}$$

The continuity of $\theta_i\left(\theta_1,\mathcal{M},\epsilon\right)$ implies that $W\left(\theta_1,\mathcal{M},\epsilon\right)$ is continuous on the compact set $[\theta^*_1-\eta,\theta^*_1+\eta]\times[0,\eta]\times\triangle^{n-1}$ and therefore uniformly continuous on that set. This implies that for all $\delta>0$ we can find $\overline{\epsilon}>0$ such that $\epsilon<\overline{\epsilon}$ implies $|W\left(\theta_1,\mathcal{M},\epsilon\right)-W\left(\theta_1,\mathcal{M},0\right)|<\delta$ for all $\mathcal{M}$ and all $\theta_1\in[\theta^*_1-\eta,\theta^*_1+\eta]$. Since $W\left(\theta_1,\mathcal{M},0\right)$ does not depend on $\mathcal{M}$, we have that for all $\delta>0$ there exists $\overline{\epsilon}>0$ such that $\epsilon<\overline{\epsilon}$ implies $|W(\theta_1,\mathcal{M}',\epsilon)-W(\theta_1,\mathcal{M},0)|<\delta$ for all $\mathcal{M},\mathcal{M}'$ and all $\theta_1\in[\theta^*_1-\eta,\theta^*_1+\eta]$.

Consider $\theta'_1\in(\theta^*_1-\eta,\theta^*_1)$, and $\theta''_1\in(\theta_1,\theta^*_1+\eta)$ such that $\theta_{n-1}\left(\theta''_1,\mathcal{M},0\right)<1$. Then condition (M) implies that

$$W\left(\theta'_1,\mathcal{M},0\right)<0<W\left(\theta''_1,\mathcal{M},0\right)$$

for all $\mathcal{M}$. Then it follows from our earlier argument that we can find $\tilde{\epsilon}>0$ such that $\epsilon<\tilde{\epsilon}$ implies that

$$W\left(\theta'_1,\mathcal{M},\epsilon\right)<0<W\left(\theta''_1,\mathcal{M},\epsilon\right)$$

for all $\mathcal{M}$. Hence, by the intermediate value theorem, for all $\epsilon<\tilde{\epsilon}$ and for all $\mathcal{M}$ there exists $\theta_1$ for which

$$W(\theta_1,\mathcal{M},\epsilon)=0.$$

It is easy to see, for this $\theta_1$, that the boundary values $\theta_1,\theta_2\left(\theta_1,\mathcal{M},\epsilon\right),\ldots,\theta_{N-1}\left(\theta_1,\mathcal{M},\epsilon\right)$ describe an equilibrium partition that is adapted to $\mathcal{M}$. ■

**Proof of Proposition 6.** Suppose that there is a separating noise equilibrium. Then by the single-crossing condition ($U^R_{12}(a,\theta)>0$), $\omega\left(\cdot\right)$ is strictly monotonic. Therefore $\omega(\cdot)$ is continuous except at a countable number of types. Let $\theta$ be a point of continuity of $\omega(\cdot)$ and suppose that $\omega(\theta) \neq \arg\max_a U^S(a,\theta,b)$. Then we can find a type $\theta'$ near $\theta$ such that both types either prefer $\omega(\theta)$ to $\omega(\theta')$ or prefer $\omega(\theta')$ to $\omega(\theta)$, violating incentive compatibility. It follows that $\omega(\theta) = \arg\max_a U^S(a,\theta,b)$ at all $\theta$ at which $\omega(\cdot)$ is continuous. For any $\eta > 0$ we can find $\theta$ such that $1-\theta < \eta$ and $\omega(\cdot)$ is continuous at $\theta$. Furthermore, we can choose $\eta$ small enough to ensure that the receiver's optimal response to the message sent by type $\theta$ is less than $\arg\max_a U^R(a,\theta)$. This implies $\omega(\theta) < \arg\max_a U^R(a,\theta) < \arg\max_a U^S(a,\theta,b) = \omega(\theta)$, establishing a contradiction. ■

**Boundary type in two-step equilibrium** (from section 3.1)

$$
\begin{aligned}
\theta_1(b,\epsilon,m_1) &= \frac{A}{6(1-\epsilon)}\cos\left(\frac{1}{3}(4\pi+\arccos(B))\right) \\
\text{where } A &= 3-4b(1-\epsilon)-6\epsilon m_1+2\sqrt{(4b(1-\epsilon))^2+3\left(1+2\epsilon-4\epsilon^2(1-m_1)m_1\right)} \\
\text{and } B &= \frac{4b(1-\epsilon)\left(9\left(1-\epsilon+2\epsilon^2(1-m_1)m_1\right)-(4b(1-\epsilon))^2\right)+27\epsilon(1-2m_1)}{\left((4b(1-\epsilon))^2+3\left(1+2\epsilon-4\epsilon^2(1-m_1)m_1\right)\right)^{\frac{3}{2}}}
\end{aligned}
$$

**Example 1,** *Three-step equilibrium*, calculation of $EU^R$
With an error of $\epsilon = \frac{1}{126}$, we showed that there is an equilibrium partition with elements $\left[0,\frac{1}{25}\right)$, $\left[\frac{1}{25},\frac{8}{25}\right)$, and $\left[\frac{8}{25},1\right]$. In the event of no error, these elements induce actions $a_1 = \frac{1}{10}$, $a_2 = \frac{9}{50}$, and $a_3 = \frac{33}{50}$ respectively. The expected utility of the receiver is given by:

$$
\begin{aligned}
EU^R &= (1-\epsilon)\left(\int_0^{\frac{1}{25}} -(\theta-a_1)^2\,d\theta + \int_{\frac{1}{25}}^{\frac{8}{25}} -(\theta-a_2)^2\,d\theta + \int_{\frac{8}{25}}^{1} -(\theta-a_3)^2\right) \\
&\quad +\epsilon\left(\int_0^1 -(\theta-a_1)^2\,d\theta\right) \\
&= -\frac{36}{1200}
\end{aligned}
$$

**Proof of Proposition 7.** Suppose that

$$
\frac{1}{2N^2} < b < \frac{1}{2(N-1)^2}
$$

for some integer $N > 1$. We show there exists an $\overline{\epsilon} > 0$ such that for all $\epsilon \in (0,\overline{\epsilon})$, there is an $N$-step $\epsilon$-noise equilibrium of the model that Pareto dominates the Pareto optimal CS equilibrium.[21]

Let the probability of error be $\epsilon$. Consider the partition $\{[0,\theta_1), \ldots, [\theta_{N-1}, 1]\}$, and suppose the sender obeys the following strategy:

$$\begin{aligned}
\text{If } \theta &\in [0,\theta_1], \text{ randomize uniformly on } [0,1] \setminus \{m_2, \ldots, m_N\} \\
\text{If } \theta &\in (\theta_1, \theta_2], \text{ send message } m_2 \\
&\vdots \\
\text{If } \theta &\in (\theta_{N-1}, 1], \text{ send message } m_N
\end{aligned}$$

The actions chosen in each step are, respectively

$$\begin{aligned}
\text{If } m &\in [0,1] \setminus \{m_2, m_3\} \text{ is received, choose } a_1 = \frac{(1-\epsilon)\,\theta_1 \frac{\theta_1}{2} + \epsilon \frac{1}{2}}{(1-\epsilon)\,\theta_1 + \epsilon} \\
\text{If } m &= m_2 \text{ is received, choose } a_2 = \frac{\theta_1 + \theta_2}{2} \\
&\vdots \\
\text{If } m &= m_2 \text{ is received, choose } a_N = \frac{\theta_{N-1} + 1}{2}
\end{aligned}$$

Solving the indifference conditions

$$\begin{aligned}
\theta_1 + b &= \frac{a_1 + a_2}{2} \\
\theta_2 + b &= \frac{a_2 + a_3}{2} \\
&\vdots \\
\theta_{N-1} + b &= \frac{a_{N-1} + a_N}{2}
\end{aligned}$$

gives

$$\theta_2 = \frac{2\theta_1 (2b + \theta_1) + \epsilon (\theta_1 - 1)(4b + 2\theta_1 - 1)}{\epsilon (\theta_1 - 1) + \theta_1} \tag{4}$$

$$\theta_3 - \theta_2 = \theta_2 - \theta_1 + 4b \tag{5}$$

$$\vdots$$

$$1 - \theta_{N-1} = \theta_{N-1} - \theta_{N-2} + 4b \tag{6}$$

[21] As an aside, it is worth noting that as $\epsilon$ tends to 0, the equilibrium constructed here tends to the most informative CS equilibrium.

Now observe that

$$\sum_{i=1}^{N-1} (\theta_{i+1} - \theta_i) = 1 - \theta_1 \tag{7}$$

and combining (5)–(6) we obtain

$$\sum_{i=1}^{N-1} (\theta_{i+1} - \theta_i) = \frac{(N-1)(N-2)}{2} 4b + (N-1)(\theta_2 - \theta_1) \tag{8}$$

(7) and (8) give us

$$\begin{aligned} 1 - \theta_1 &= \frac{N(N-1)}{2} 4b + (N-1)(\theta_2 - \theta_1) \\ \Rightarrow \theta_2 - \theta_1 &= \frac{1-\theta_1}{N-1} - 2b(N-2) \end{aligned} \tag{9}$$

Finally, solving (4) and (9) for $\theta_1$, we find

$$\begin{aligned} \theta_1 &= \frac{1 + 2bN - 2\epsilon N - 2b\epsilon N - 2bN^2 + 2b\epsilon N^2}{2(1-\epsilon)N} \\ &+ \frac{\sqrt{4\epsilon(\epsilon-1)(-1+2b(N-1))N^2 + (1 - 2\epsilon N + 2b(\epsilon-1)(N-1)N)^2}}{2(1-\epsilon)N} \end{aligned} \tag{10}$$

(4) gives us the position of the first boundary point, $\theta_1$, in an $N$ step $\epsilon$-noise equilibrium. Solving for $\epsilon$, we can find the level of noise required to sustain a particular equilibrium value of $\theta_1$:

$$\epsilon = \frac{\theta_1 (-1 - 2bN + 2bN^2 + N\theta_1)}{N(1-\theta_1)(1 + 2b - 2bN - \theta_1)} \tag{11}$$

The expected utility of receiver is given by

$$\begin{aligned} EU^R_{noise} &= (1-\epsilon)\left(\int_0^{\theta_1} -(\theta - a_1)^2 d\theta + \int_{\theta_1}^{\theta_2} -(\theta - a_2)^2 d\theta + \cdots + \int_{\theta_{N-1}}^{1} -(\theta - a_N)^2\right) \\ &+\epsilon\left(\int_0^1 -(\theta - a_1)^2 d\theta\right) \\ &= (1-\epsilon)\left(-\frac{\theta_1\left(-2\epsilon(\theta_1 - 1)\theta_1^3 + \theta_1^4 + \epsilon^2(\theta_1 - 1)^2\left(3 + \theta_1^2\right)\right)}{12(\epsilon + \theta_1 - \epsilon\theta_1)^2} - \frac{1}{12}\sum_2^N (\theta_i - \theta_{i-1})^3\right) \\ &+\epsilon\left(-\frac{\epsilon^2 - 2(\epsilon-1)\epsilon\theta_1 + 4(\epsilon-1)^2\theta_1^2 - 6(\epsilon-1)^2\theta_1^3 + 3(\epsilon-1)^2\theta_1^4}{12(\epsilon + \theta_1 - \epsilon\theta_1)^2}\right) \end{aligned}$$

Solving and substituting for $\epsilon$ using (11), we can re-write the expected utility of the receiver in terms of $\theta_1$:

$$EU^R_{noise} = -\frac{4b^2 (N-2)(N-1)^2 N + 4b(N-1)^2 (2N-1)\theta_1 + ((2N-1)\theta_1 - 1)^2}{12(N-1)^2}$$

To see that this equilibrium Pareto dominates the Pareto optimal CS equilibrium for small $\epsilon$, we consider two cases.

**Case 1:**

$$\frac{1}{2N(N-1)} \leqslant b < \frac{1}{2(N-1)^2}$$

The Pareto optimal CS equilibrium has $N-1$ steps, with resulting expected utility

$$EU^R_{CS} = -\frac{1}{12(N-1)^2} - \frac{b^2\left((N-1)^2 - 1\right)}{3}$$

Notice that this equilibrium coincides precisely with the construction above when $\epsilon = 0 \Rightarrow \theta_1 = 0$. By introducing a small amount of noise, we are able to squeeze an extra step into the equilibrium partition. We now compute the difference in the receiver's expected utility in the two types of equilibrium.

$$\begin{aligned}
& EU^R_{noise} - EU^R_{CS} \\
= & -\frac{4b^2 (N-2)(N-1)^2 N + 4b(N-1)^2 (2N-1)\theta_1 + ((2N-1)\theta_1 - 1)^2}{12(N-1)^2} \\
& -\left(-\frac{1}{12(N-1)^2} - \frac{b^2\left((N-1)^2 - 1\right)}{3}\right) \\
= & -\frac{(2N-1)\theta_1\left(4b(N-1)^2 - 2 + (2N-1)\theta_1\right)}{12(N-1)^2} \\
> & \ 0 \text{ for } \theta_1 \in \left(0, \frac{2 - 4b(N-1)^2}{2N-1}\right)
\end{aligned}$$

Substituting for $\theta_1$, we obtain $EU^R_{noise} - EU^R_{CS} > 0$ if

$$\epsilon \in \left(0, \frac{2\left(1 - 2b(N-1)^2\right)(1 + 2b(N-1)N)}{\left((2N-3)^2 + 2b(2N-3)^2(N-1) - 8b^2(N-1)^3\right)N}\right)$$

To see that the upper bound of this interval is strictly positive, we show that both numerator and denominator are strictly positive. Consider first the numerator. Clearly $(1 + 2b(N-1)N) > 0$, and since $b < \frac{1}{2(N-1)^2}$, we also have $\left(1 - 2b(N-1)^2\right) > 0$ as required. For the denominator, suppose first that $N = 2$ (recall that $N$ is an integer greater than 1). Then the denominator simplifies to $2(1 - 2b(4b - 1)) > 0$ since $b < \frac{1}{2}$. Now suppose $N \geqslant 3$. We can rewrite the denominator as follows:

$$\begin{aligned} & N\left((2N-3)^2 + 2b(2N-3)^2(N-1) - 4b(N-1)2b(N-1)^2\right) \\ > \ & N\left((2N-3)^2 + 2b(2N-3)^2(N-1) - 4b(N-1)\right) \\ = \ & N\left((2N-3)^2 + 2b\left((2N-3)^2 - 2\right)(N-1)\right) \\ > \ & 0 \text{ as required.} \end{aligned}$$

**Case 2:**

$$\frac{1}{2N^2} < b < \frac{1}{2N(N-1)}$$

In this case, the most informative CS equilibrium has $N$ steps, with resulting expected utility

$$EU_{CS}^{R} = -\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3}$$

This equilibrium coincides precisely with the construction above when $\epsilon = 0$ and $\theta_1 = \frac{1-2b(N-1)N}{N}$. By introducing a small amount of noise, we increase the size of the first (and smallest) element of the equilibrium partition. As before, we compute $EU_{noise}^{R} - EU_{CS}^{R}$.

$$\begin{aligned} & EU_{noise}^{R} - EU_{CS}^{R} \\ = \ & -\frac{4b^2(N-2)(N-1)^2 N + 4b(N-1)^2(2N-1)\theta_1 + ((2N-1)\theta_1 - 1)^2}{12(N-1)^2} \\ & -\frac{1}{12N^2} - \frac{b^2(N^2-1)}{3} \\ = \ & \frac{(2N-1)\left(-1 + 4b^2(N-1)^2 N^2 - 4b(N-1)^2 N^2\theta_1 - 2N^3\theta_1^2 + N^2\theta_1(2+\theta_1)\right)}{12(N-1)^2 N^2} \\ > \ & 0 \text{ for } \theta_1 \in \left(\frac{1-2bN(N-1)}{N}, \frac{1+2bN(N-1)}{N(2N-1)}\right) \end{aligned}$$

Substituting for $\theta_1$, we obtain $EU_{noise}^{R} - EU_{CS}^{R} > 0$ if

$$\epsilon \in \left(0, \frac{2(1+2b(N-1)N)(2bN^2-1)}{(N-1)(1+2(1-b)N)(1+2N-4bN^2)}\right)$$

It is easy to see that the upper bound is strictly positive, completing the proof.■

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