Single or Multiple Pricing in Electricity Pools?

Ahmed W Anwar (University of Edinburgh)

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A.W. Anwar
Department of Economics
University of Edinburgh
WRB, 50 George Square
Edinburgh
EH8 9JY

e-mail: Ahmed.Anwar@ed.ac.uk

Abstract

We present a 2 bidder multi-unit, common cost auction model with uncertain demand and capacity constraints which ensure that the participants sometimes face a residual market share. The model is motivated by electricity pools. We show that a single-price auction where the bidders can submit only one bid for all units weakly dominates an auction where the bidders can make multiple-price bids in terms of average prices. In the case of uniform price auctions we give an example where the dominance is strict.

1 Introduction

Since 1990, electricity markets around the world have gone through a radical transformation. The benchmark for change was the England and Wales reform which separated generation from transmission. The rationale behind a vertically integrated supply structure is the need for a centralized system which ensures supply and demand are constantly matched at minimum cost subject to network constraints. This requires a central dispatcher to constantly monitor the system and instruct units when to switch on and off. However, a major drawback of an integrated system is that there are no competitive market forces acting on the potentially competitive generating system. At the heart of the England and Wales reform was the innovative way in which a competitive generating system was combined with a centralized dispatch system. This was done by setting up an electricity pool which is a spot market for the sale of electricity. Generators compete to supply electricity by submitting bids for the minimum price at which they are willing to supply from each of their plants. The central dispatcher then constructs the least cost rank order of plants for each period (half an hour in the England and Wales pool). Hence the mechanism that determines which units are dispatched is a multi-unit auction. A uniform-price auction was initially adopted in the England and Wales pool where the units dispatched are paid the marginal price (the bid on the marginal unit). Similar systems have been adopted and are under consideration elsewhere[1]. The various systems differ in the way they deal with such things as imbalances, network constraints and incentives for investment. In this paper, we restrict attention to the multi-unit auction aspect of electricity pools, a feature they all have in common.

The England and Wales reform has not been without problems. The hope that this market structure would lead to competition in generation was not realized with average prices well above competitive levels[2]. Perhaps the biggest mistake was that the thermal plants were divided between only two companies which gave the two large generators significant market power.

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[1] Competitive electricity pools are currently being used in Australia, California, Norway and Sweden. The international experience is reviewed by von der Fehr and Harbord [9].

[2] See for example Wolak and Patrick [15].

This has been the focus of the theoretical analysis of the England and Wales pool. Green and Newbery [10][3] and von der Fehr and Harbord [8][4] show that splitting the thermal generators between more than two companies would have led to lower pool prices.

In this paper we model a 2 bidder multi-unit, common cost auction with uncertain demand and capacity constraints which ensure that the participants sometimes face a residual market share. We demonstrate a revenue ranking between types of equilibria, flat-supply where all units are bid at the same price and increasing-supply where they are not. We show that flat-supply equilibria dominate increasing-supply equilibria in terms of average prices. This ranking demonstrates that allowing the bidders to submit multiple price bids can only increase the average price since with single price bids they are forced into a flat-supply equilibrium.

We also look at a comparison between pricing rules when multiple price bids are allowed. An alternative to the uniform price rule is the discriminatory price rule where the generators are paid their actual bids for units dispatched. We identify a tendency to differentiate prices under a uniform-price rule (by submitting increasing-supply functions) and to set uniform prices under a discriminatory auction (by submitting flat-supply functions). Intuitively, the strategies used counterbalance the price rule. We give an example (with uniform demand distribution) of a uniform price auction equilibrium with multiple price bids that does strictly worse than the single price (flat supply) equilibrium. However, a complete ranking is not possible in the general case as we do not demonstrate that there are no equilibria of the discriminatory auction that do strictly worse.

These results are of general theoretical interest. Revenue rankings have proved hard to come by in the multi-unit auction case when each bidder demands more than one unit. Much of the interest in multi-unit auctions stems from the sale of Treasury bonds where both discriminatory and uniform auc-

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[3]Green and Newbery use the supply function framework of Klemperer and Meyer [11]. They simulate the England and Wales spot market to measure the extent and cost of market power. They demonstrate that if the thermal plants had been split between five companies rather than two, equilibrium pool prices would be much closer to competitive levels.

[4]Von der Fehr and Harbord model the pool as a uniform-price, multi-unit auction. They also find that the expected pool price is lower in a more fragmented industry.

tions have been used. There is a significant amount of empirical work comparing the two auctions but this has proved inconclusive[5]. Back and Zender [2] use the share auction framework of Wilson [14], where the good is perfectly divisible and has a common value. They find that with a uniform auction, any price between the reservation price and the lower bound of the common-value distribution can be supported as a symmetric Nash equilibrium. There is therefore a multiple-equilibrium problem and some of these equilibria are extremely bad for the seller. However, the equilibria they construct do not hold in the case where indivisible units are for sale. Ausubel and Cramton [1] concentrate on the relative efficiency of the auctions but also demonstrate that in the flat demand case, the discriminatory equilibrium revenue dominates the uniform-price equilibrium. Engelbrecht-Wiggans and Kahn [5] and Noussair [12] look at discrete private value, multi-unit, uniform-price auctions where the bidders demand two units. They find that the uniform auction possesses equilibria that are extremely bad for the seller because the bidders will shade their second bids substantially as this reduces the price they pay for the first unit when the second bid is marginal. Engelbrecht-Wiggans and Kahn [6] look at the two unit case of the discriminatory auction. They find equilibria that are better than the bad equilibria of the uniform auction. However, a revenue comparison is not possible as there may be other equilibria of both auctions.

There are two features of our model that distinguish it from this multi-unit literature. We have uncertain demand and a binding capacity constraint which ensures that the firms face some residual market. Perhaps the best example of such a market in practice is an electricity pool where the uncertainty arises from the fact that the bids are made before demand is realised and the constraints reflect the generating capacity of each firm. Such constraints should be present in any efficient generating system as if it is the case that there are m firms and demand can always be met by $m - 1$ firms then there is surely over-investment in the generating stock. Fabra *et al.* [7] look at the multi-unit case with certain demand and show that the uniform auction leads to the worst possible outcome for the buyer. However, these pure-strategy equilibria do not hold when there is uncertain demand. They also look at

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[5]Binmore and Swierzbinski [3] review the empirical evidence.

the single-unit case with uncertain demand and show that the expected costs are the same under the two auctions. This result does not go through to the multi-unit case because the uniform-price auction with multiple price bids has equilibria where the firms mix between increasing supply functions. We look at the multi-unit case with uncertain demand and show that the unique equilibrium of the single price auction weakly dominates any other equilibrium in terms of average price. In the case of uniform-price auctions we show that there are equilibria that do strictly worse when multiple price bids are allowed. In section 2 we look at a simple example to highlight some of the intuition underlying the main results. Section 3 presents the general model.

2   A simple example

Consider 2 suppliers that can supply up to 2 units each. They compete by submitting a bid if a single price auction is used and two bids if a multiple price auction is used. The marginal cost of producing each unit is c and the buyer sets a maximum permissible bid (*reserve price*) $\overline{p}$ per unit. After bids are submitted, nature chooses the level of demand, d, from the set $[1, 2, 3, 4]$ and the lowest d bids win. Let q^i be the probability that demand is i.

The cases where demand is certain to be no more than the capacity of each firm ($q^3 = q^4 = 0$) and where demand is certain to be greater than the capacity of each firm ($q^1 = q^2 = 0$) are covered by Fabra *et al* [7]. In the first case, the capacity constraint is not *binding* as demand is never greater than 2. In the second case, the capacity constraint is always binding as each firm will have a residual market share irrespective of how they bid.

In a uniform-price auction all successful bids are paid at the marginal price. Without a binding capacity constraint, Bertrand type competition ensures that there is no equilibrium where the units sell for a price greater than c. However, when the capacity constraint is always binding, we get the other extreme result where all the units sell for the maximum permissible price $\overline{p}$. The equilibrium involves one firm submitting two bids at $\overline{p}$ and the other submitting low bids. Since the bids of the low price firm are not marginal, there is no incentive to raise them and if the bids are sufficiently low, say at marginal cost, there is no gain to be had from undercutting by the high price firm.

An alternative to the uniform-price auction is the discriminatory-price auction where all successful units are simply paid at the actual bids. This does not alter the competitive outcome when there is no binding capacity constraint. Again, as each firm can supply the entire market, Bertrand type competition ensures that there is no equilibrium where the units sell for a price greater than c. However, the equilibria are different in the second case $(q^1 = q^2 = 0)$. We no longer get the extreme result that all units sell for $\overline{p}$. This would require both firms to submit each unit at $\overline{p}$ which cannot be an equilibrium as each firm would have an incentive to undercut slightly and avoid rationing when demand is 3. This simple analysis suggests that the buyer should use the discriminatory auction in preference to the uniform auction when there is a binding capacity constraint as the price paid in equilibrium will be less than $\overline{p}$. The strong intuition here is that with the uniform-price rule, it is only the marginal price that matters in determining the payment for all the successful units. If one firm submits its capacity at a low price, there is no competition over this marginal unit which will then be set at the highest permissible price. With the discriminatory-price rule however, there is an incentive to increase all bids to increase payment but at the same time to reduce high bids that are set to be unsuccessful. This ensures that there is always competition between all the units.

This ranking is limited however, as it only applies when the opponent's capacity is always less than demand. In practice, when demand is uncertain, it may be possible that a binding capacity constraint arises with a positive probability but not with certainty. That is, there may be a high realization of demand that results in a binding capacity constraint or there may be a low realization of demand such that each firm can supply the entire market. This removes the extreme result in the uniform auction case where all units sell at $\overline{p}$ in equilibrium. The reason the equilibrium described above breaks down is that the bids of the low price firm now set the marginal price when demand in low which gives the firm an incentive to increase these bids. Once they have been increased sufficiently, the high price firm can gain by undercutting. In fact, when all realizations of demand occur with a positive probability, there is no pure-strategy equilibrium in either auction. To rank the auctions in this more general case, it is therefore necessary to look at the mixed-strategy equilibria. We show that the single price and multiple price cases no longer

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give rise to the same equilibria if a uniform-price auction is used. To see this, consider demand that is uniformly distributed $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$. In either single price case (discriminatory or uniform) there is a unique equilibrium where the firms randomly select a bid from $p\epsilon[v^i, \overline{p}]$ (where $i\epsilon(D, U)$) according to some distribution function $F(p)$. The distribution F has no mass points.

In this (flat-supply) equilibrium both firms are indifferent between offer prices $p\epsilon[v^i, \overline{p}]$. Setting $\overline{p}$ will mean that both units are certain to be ranked below the opponent's as there are no mass points and in the event they are successful they sell at $\overline{p}$. The expected profit of each supplier is therefore $(1 \times \frac{1}{4} + 2 \times \frac{1}{4})(\overline{p} - c) = \frac{3}{4}(\overline{p} - c)$. Each firm can make at least this much profit by deviating from any increasing-supply equilibrium and setting both prices at $\overline{p}$. If there are no mass points at $\overline{p}$, then the deviation will yield profit $\frac{3}{4}(\overline{p} - c)$. If the opponent does have mass points at $\overline{p}$ then the deviation will be worth even more as there is a positive probability of selling units when demand is 1 or 2. Now since the marginal cost is constant, all outcomes are equally efficient. To rank the auctions it is therefore sufficient to look at the profits of the suppliers which by the above argument cannot be less than $\frac{3}{4}(\overline{p} - c)$ in any equilibrium. Hence expected profits from any increasing-supply equilibrium are greater than or equal to expected profits from the flat-supply equilibrium.

In the multiple-price discriminatory auction the equilibrium remains the same. In the multiple-price uniform auction case there is an equilibrium where each firm chooses a price $p_1\epsilon[v_1, v_2]$ according to some distribution function $F_1(p_1)$ for the first unit and $p_2\epsilon[v_2, \overline{p}]$ according to some distribution function $F_2(p_2)$ for the second unit. This is an increasing-supply equilibrium where the expected profits are strictly greater than $\frac{3}{4}(\overline{p} - c)$.

With this uniform demand distribution, there is no flat-supply equilibrium in the uniform auction. This contrast with the discriminatory auction is important. To see the intuition behind it consider the gain from differentiating prices in the uniform auction relative to the discriminatory case. First consider lowering the price of the first (lower priced) unit. This will increase the probability the unit is sold under both rules but the payment under a uniform rule will only decrease when the unit is marginal. With a discriminatory rule, the payment will be reduced even when demand is high and the bid is certain to be successful. Now consider increasing the price of the second

(higher priced) unit. This will reduce the probability the unit is sold and increase the price of the unit under both rules. However, when demand is high, the price of the first unit will also increase under the uniform-price rule. Hence for any given strategy the opponent is using, differentiating prices is more attractive under the uniform price rule. This difference between the two auctions has been identified in other multi-unit contexts. Engelbrecht-Wiggans and Kahn [6] characterise equilibria of the discriminatory auction in the two unit case and show that there is a positive probability that a bidder bids the same for both units even when the marginal valuations are different. Ausubel and Cramton [1], Engelbrecht-Wiggans and Kahn [5] and Noussair [12] demonstrate a general incentive to differentiate bids in a uniform auction. The intuition is that the highest bids are very high to ensure they are successful (in the multi-unit demand case) and the lowest bids are very low as there is a significant probability that these bids will be marginal and therefore determine the price of all the units won.

The results demonstrate that there is never an advantage to allowing multiple-price bids. In general, if multiple price bids are allowed then the discriminatory auction has an equilibrium that is identical to the single-price equilibrium and the uniform price auction may have equilibria that result in strictly lower prices. However, we are not able to rule out increasing supply equilibria of the multiple-price discriminatory auction and so a ranking of discriminatory and uniform price equilibrium is not possible.

3   Model

There are 2 firms who can supply up to k units each at a constant marginal cost, c per unit. The firms submit bids for the minimum price at which they are willing to supply each of their units in the multiple price case and a single bid for the minimum price at which they are willing to supply all of their units in the single price case. The maximum permissible bid is $\overline{p}$ per unit. Let $S(p)$ be the number of units bid at or below p. After the firms submit their bids, nature chooses the level of demand, d. Demand is distributed on the interval $[\underline{d}, \overline{d}]$ and the distribution is common knowledge. The market clearing price, $p(d)$, is the lowest price such that $S(p) \geq d$. If more than one firm has bid units at $p(d)$ and $S(p(d)) > d$ then units bid at $p(d)$ are

rationed[6]. When demand is not an integer, we assume a fraction of a unit is sold at the margin[7]. The firms are risk-neutral. Under a uniform-price rule the sellers are paid the market clearing price, $p(d)$, for the units sold whereas they are paid the bid prices on units sold if a discriminatory rule is used.

3.1 Preliminary results

The focus of the paper is the case where demand has positive density over some interval, $[\underline{d}, \overline{d}]$ where $\overline{d} > k > \underline{d}$. That is, there is a positive probability that demand will be greater than the capacity of each firm so that each firm faces some residual market share. The cases where $k < \underline{d}$ and $\overline{d} < k$ are considered by Fabra *et al.* [7]. To summarise the results, in the case where $\overline{d} < k$ (each firm is certain to be able to supply the entire market) we get the competitive outcome in any equilibrium of the uniform or discriminatory auction. Intuitively, Bertrand type competition ensures that there is no equilibrium where the marginal price is above c. If $\overline{d} > k$ then there is no pure-strategy equilibrium of the discriminatory auction. Furthermore, in any pure-strategy equilibrium of the uniform auction the marginal price is $\overline{p}$ which gives the worst possible outcome as all units sell for the maximum permissible price. If $\overline{d} > k > \underline{d}$ and there is positive density everywhere on the interval $[\underline{d}, \overline{d}]$ then the uniform auction also possesses only mixed-strategy equilibria. These results do not depend on whether a single price or multiple price rule is used[8].

3.2 Equilibria

For the remainder of this section we assume $\overline{d} > k > \underline{d}$. From the preliminary results, there are no pure-strategy equilibria of the single or multiple price, discriminatory or uniform auctions. We therefore look for mixed-strategy equilibria. We divide all the possible equilibria into two types, flat-supply equilibria and increasing-supply equilibria.

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[6]We assume any rationing rule where all these units have a positive probability of being successful.

[7]The case where the entire unit is sold is covered by this assumption as we can simply make the demand distribution discrete.

[8]Fabra *et al* [7] consider both single-unit and multi-unit auctions. The single unit auction is equivalent to a multi-unit single price auction.

Definition 1 *A flat-supply equilibrium is a mixed-strategy equilibrium where all units are set at the same price.*

Definition 2 *An increasing-supply equilibrium is a mixed-strategy equilibrium where firm i chooses k prices according to some joint density function and the prices are different with positive probability.*

Claim 1 *In any flat-supply equilibrium (of the discriminatory or uniform auction) each firm must bid according to a density function that is strictly positive over the same atomless support $p \in (v^i, \overline{p})$, (where $i \in (D, U)$) and the expected profit of each firm is*

$$\Pi^F = (\overline{p} - c) \Pr(d > k)(E(d \mid d > k) - k).$$

Proof. In any mixed-strategy flat-supply equilibrium each firm must be indifferent between prices in the support of the equilibrium and weakly prefer not to set a price that is not in the support. Note first that this implies the lower bound of the distributions must be the same. If firm 1 has a higher lower bound, then firm 2 will strictly prefer setting a price above its lower bound as this will increase the payment received without affecting the probability the units are sold (with a uniform price auction this will happen with positive probability). Let the common lower bound be v^i.

No interior mass points: If one firm sets a price $p < \overline{p}$ with positive probability then the other firm will not set prices on $[p, p + \varepsilon]$ as bidding slightly below p increases the rank without significantly affecting the price for ε sufficiently small. However, this would imply that the firm setting p can do strictly better by setting a price just above p. Hence setting p with positive probability can not be part of a mixed-strategy equilibrium.

No holes: It is not possible to have an equilibrium where firm i does not set prices on some interval $[p_a, p_b]$ $p_a > v^i$, $p_b \leqslant \overline{p}$ if there is positive density over $(p_a - \varepsilon, p_a)$. If there is such an equilibrium, firm j will not set a price on (p_a, p_b) as increasing the price towards p_b increases the payment without affecting the probability of winning (with a uniform price auction this will happen with positive probability). However, firm i will then prefer setting a price just below p_b to anything on the interval $(p_a - \varepsilon, p_a)$ as this does not significantly affect the probability of winning but increases the payment.

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The upper bounds must be $\overline{p}$: If by contradiction one firm has upper bound $\widetilde{p}$ and the other has upper bound $\widehat{p}$ where $\overline{p} \geqslant \widehat{p} \geqslant \widetilde{p}$ and $\overline{p} > \widetilde{p}$ then the firm with the upper bound $\widehat{p}$ will strictly prefer setting $\overline{p}$ to $\widetilde{p}$ as the payment is increased without affecting the probability. This however implies there is a hole in this player's support. Hence $\widetilde{p} = \overline{p}$.

Since both firms have a common lower bound, they must make the same profit. It is not possible for both firms to have a mass point at $\overline{p}$ in equilibrium as each firm can gain by bidding $\overline{p} - \varepsilon$. Suppose firm i does not have a mass point at $\overline{p}$. Then from the above $\overline{p}$ is in the support of j's strategy and setting $\overline{p}$ will result in no units sold when $d < k$ and $(E(d \mid d > k) - k)$ on average when $d > k$ which gives expected profit $(\overline{p} - c) \Pr(d > k)(E(d \mid d > k) - k)$. Since both firms have a common lower bound, firm i must make the same profit. This implies that firm j cannot have a mass point at $\overline{p}$. ■

Corollary 1 *The expected profit in a single-price discriminatory or uniform auction must be Π^F.*

This follows from the fact that a single price auction can have only flat-supply equilibria.

Proposition 1 *The average price in any increasing supply equilibrium is at least as high as in a flat-supply equilibrium.*

Proof. The expected cost to the buyer is equal to the sum of expected profits plus the marginal cost of the units. To compare average prices in the different equilibria it is therefore sufficient to compare expected profits.

Let the profit from some increasing supply equilibrium be Π^I. If setting all units at $\overline{p}$ is in the support of the increasing-supply equilibrium strategy then $\Pi^I = \Pi^F$. If it is not then one possible deviation is to set all units at $\overline{p}$. If there are no mass points at $\overline{p}$ in the increasing-supply equilibrium, then the payoff from this deviation is Π^F. If there are mass points, the payoff will be even greater as the firm will sell more units on average at the same price. Hence $\Pi^I \geqslant \Pi^F$. QED. ■

The intuition for the result is very simple. Any bidder can obtain a payoff of at least Π^F by setting all units at $\overline{p}$. Corollary 1 shows that they cannot get more than this in a single-price auction. We now show that if multiple-price

bids are allowed then there is generally an equilibrium of the discriminatory auction that gives the single-price payoffs.

Proposition 2 *There is a flat-supply equilibrium under a discriminatory rule for any demand distribution that satisfies*

$$q^d(d^d - (2k - \alpha))q^a d^a \quad > \quad (q^c + q^d)(q^d(2k - d^d)(k - \alpha) + q^c(d^c - k)\alpha) \qquad (1)$$
$$+ q^b(d^b - (k - \alpha))q^c(d^c - k) + q^b q^d(k - \alpha)(d^b - d^d + k)$$

where $q^a = Pr(0 < d \leqslant k - \alpha)$, $q^b = Pr(k - \alpha < d \leqslant k)$, $q^c = Pr(k < d \leqslant 2k - \alpha)$, $q^d = Pr(2k - \alpha < d \leqslant 2k)$, $d^a = E(d \mid 0 < d \leqslant k - \alpha)$, $d^b = E(d \mid k - \alpha < d \leqslant k)$, $d^c = E(d \mid k < d \leqslant 2k - \alpha)$, $d^d = E(d \mid 2k - \alpha < d \leqslant 2k)$ for all $\alpha \epsilon [1, k - 1]$.

Finally we characterise equilibria of the uniform auction when the demand distribution is uniform on the interval $[0, 2k]$.

Proposition 3 *Under a uniform-price rule there is no flat-supply equilibrium when demand is uniformly distributed on the interval $[0, 2k]$.*

An equilibrium in the uniform price case must then involve mixing between increasing supply functions. Increasing-supply equilibria are in general difficult to characterize as they involve each firm choosing prices according to some joint density. We have however managed to characterize a particular type of increasing-supply equilibrium that exploits the tendency to submit different prices.

Definition 3 *An independent increasing-supply equilibrium is an equilibrium where the firms choose k prices independently, the first on some interval $[v_1, v_2]$ according to some density f_1, the second on some interval $[v_2, v_3]$ according to some density $f_2 \dots$ and the kth on some interval $[v_k, v_{k+1}]$ according to some density f_k.*

Proposition 4 *Under a uniform-price rule there is an independent increasing-supply equilibrium when demand is uniformly distributed on the interval $[0, 2k]$ which leads to an average price that is greater than in the single-price equilibrium.*

Allowing multiple price bids can never result in lower average prices and at least in the case of a uniform price auction it can result in higher average prices. We therefore conclude that in a multi-unit auction with common costs, a single-price rule should be used.

A Appendix

Proposition 5 *There is a flat-supply equilibrium under a discriminatory rule for any demand distribution that satisfies*

$$q^d(d^d - (2k - \alpha))q^a d^a > (q^c + q^d)(q^d(2k - d^d)(k - \alpha) + q^c(d^c - k)\alpha) \tag{2}$$
$$+q^b(d^b - (k - \alpha))q^c(d^c - k) + q^b q^d(k - \alpha)(d^b - d^d + k)$$

where $q^a = Pr(0 < d \leqslant k - \alpha)$, $q^b = Pr(k - \alpha < d \leqslant k)$, $q^c = Pr(k < d \leqslant 2k - \alpha)$, $q^d = Pr(2k - \alpha < d \leqslant 2k)$, $d^a = E(d \mid 0 < d \leqslant k - \alpha)$, $d^b = E(d \mid k - \alpha < d \leqslant k)$, $d^c = E(d \mid k < d \leqslant 2k - \alpha)$, $d^d = E(d \mid 2k - \alpha < d \leqslant 2k)$ for all $\alpha \epsilon [1, k - 1]$.

Proof. Suppose firm 2 is submitting a price for all units, according to the distribution function $F(p)$. Let $f(p)$ be the corresponding density function. From claim 1, the support must be $[v^D, \overline{p}]$. Player 1's expected payoff from submitting a price p_1 for all units is

$$\Pi(p_1) = q^h \left(\int_{p_1}^{\overline{p}} (p_1 - c) k f(p) dp + \int_{v^D}^{p_1} (p_1 - c)(d^h - k) f(p) dp \right)$$
$$+ q^l \int_{p_1}^{\overline{p}} (p_1 - c) d^l f(p) dp$$

where $q^h = \text{Prob}(d > k)$, $d^h = E(d \mid d > k)$, $q^l = \text{Prob}(d < k)$, $d^l = E(d \mid d < k)$.

In equilibrium $\Pi'(p_1) = 0$ for all $p_1 \in (v, \overline{p})$. This gives

$$(p - c)f(p) + F(p) = \frac{kq^h + q^l d^l}{q^h(2k - d^h) + q^l d^l} \tag{3}$$

From claim 1, in any flat-supply equilibrium each firm must bid according to a density function that is strictly positive over the same support $p \in [v^D, \overline{p}]$ with no mass points. The distribution function that both firms use in a

flat-supply equilibrium must therefore satisfy (3) and it must be the case that $F(\overline{p}) = 1$. The unique solution of this differential equation with the boundary condition $F(\overline{p}) = 1$ is

$$f(p) = \frac{(d^h - k)q^h}{q^h(2k - d^h) + q^l d^l} \frac{(\overline{p} - c)}{(p - c)^2} \tag{4}$$

$$F(p) = \frac{(kq^h + q^l d^l)p}{(q^h(2k - d^h) + q^l d^l)(p - c)} - \frac{(d^h - k)q^h \overline{p} + (q^h(2k - d^h) + q^l d^l)c}{(q^h(2k - d^h) + q^l d^l)(p - c)} \tag{5}$$

Solving $F(v^D) = 0$ gives $v^D = ((d^h - k)q^h \overline{p} + (q^h(2k - d^h) + q^l d^l)c)/(kq^h + q^l d^l)$. We therefore have an equilibrium where both firms mix using (5).

Now we need to check that it is globally optimal to set a single price. Let P be a vector of n prices, $\{p_1, ...p_n\}$ where $n \leq k$, $p_1 < p_2.... < p_n$, $p_n \leq \overline{p}$, $p_1 \geq v$[9]. Let α be the number of units bid at p_n, $q^a = Pr(0 < d \leqslant k - \alpha)$, $q^b = Pr(k - \alpha < d \leqslant k)$, $q^c = Pr(k < d \leqslant 2k - \alpha)$, $q^d = Pr(2k - \alpha < d \leqslant 2k)$, $d^a = E(d \mid 0 < d \leqslant k - \alpha)$, $d^b = E(d \mid k - \alpha < d \leqslant k)$, $d^c = E(d \mid k < d \leqslant 2k - \alpha)$, $d^d = E(d \mid 2k - \alpha < d \leqslant 2k)$. The profit made on the units priced at p_n is independent of all the other prices[10]. It is

$$
\begin{aligned}
\Pi_n(p_n) \;=\; & q^b \int_{p_n}^{\overline{p}} (p_n - c)(d^b - (k - \alpha))f(p)dp \\
& + q^c \int_{p_n}^{\overline{p}} (p_n - c)\alpha f(p)dp \\
& + q^d \left(\begin{array}{c} \int_{p_n}^{\overline{p}} (p_n - c)\alpha f(p)dp + \\ \int_{v^D}^{p_n} (p_n - c)(d^d - (2k - \alpha))f(p)dp \end{array} \right)
\end{aligned}
$$

Substituting for $f(p)$ using (4) and simplifying, the first derivative of the profit function with respect to p_n is,

$$
\begin{aligned}
\frac{\partial \Pi_n(p_n)}{\partial p_n} \;=\; & -\frac{(q^c + q^d)(q^d(2k - d^d)(k - \alpha) + q^c(d^c - k)\alpha)}{(q^h(2k - d^h) + q^l d^l)} \\
& -\frac{(d^b - (k - \alpha))q^b(q^d(d^d - k) + q^c(d^c - k))}{(q^h(2k - d^h) + q^l d^l)} \\
& +\frac{q^d(d^d - (2k - \alpha))(q^a d^a + q^b d^b)}{(q^h(2k - d^h) + q^l d^l)}
\end{aligned} \tag{6}
$$

---

[9] Any bid below v cannot be optimal as firm 1 will be assigned such units with probability 1 but can increase the payment received for these units by increasing the bid to v.

[10] The other prices affect neither the payment nor the rank of the units priced at p_n.

Hence, the first derivative is a constant and if it is negative, the firm can always gain by decreasing the highest price towards the second highest. If it is negative for all α between 0 and k, then it is optimal to submit a flat supply function. Consider the case where demand is always greater than k, $q^a = q^b = 0$, $q^c + q^d = 1$. The derivative is then

$$-\frac{(q^d(2k - d^d)(k - \alpha) + q^c(d^c - k)\alpha)}{(q^h(2k - d^h) + q^l d^l)}$$

which is always negative. The flat-supply equilibrium therefore holds for any demand distribution where $\underline{d} > k$. Next consider the case where $d \sim U[0, 2k]$. Then $q^a = (k - \alpha)/2k$, $q^b = a/2k$, $q^c = (k - a)/2k$, $q^d = a/2k$, $d^a = (k - a)/2$, $d^b = (2k - a)/2$, $d^c = (3k - a)/2$, $d^d = (4k - a)/2$. Substituting these values into (6) gives,

$$-\frac{1}{4}\frac{\alpha(k - \alpha)}{k}$$

Again, this is always negative. The general condition for the equilibrium not to hold is,

$$q^d(d^d - (2k - \alpha))q^a d^a \;>\; (q^c + q^d)(q^d(2k - d^d)(k - \alpha) + q^c(d^c - k)\alpha) \qquad (7)$$
$$+ q^b(d^b - (k - \alpha))q^c(d^c - k) + q^b q^d(k - \alpha)(d^b - d^d + k)$$

As α increases $q^a d^a$ decreases but $q^d(d^d - (2k - \alpha))$ increases. This puts a limit on the values the left hand side of (7) can take.

It is clear from (7) that for the equilibrium not to hold, the demand distribution must be skewed to the right so that $q^a d^a$ is significant but with sufficient mass on the other tail. Hence for most distributions the equilibrium will hold. However, it is possible to construct exceptions. To see this consider the case where $d^a = (k - a)/2$, $d^b = (2k - a)/2$, $d^c = (3k - a)/2$, $d^d = (4k - a)/2$. Substituting these values into the numerator of (6) gives,

$$-\frac{1}{4}\alpha(k - \alpha)(2(q^d)^2 + q^c(2q^c + q^b) + q^d(4q^c - q^a))$$

This expression is greater than zero when,

$$q^a > \frac{q^c(2q^c + q^b)}{q^d} + 4q^c + 2q^d \qquad (8)$$

For example, if $q^a = .85$, $q^b = .05$, $q^c = .05$ and $q^d = .05$ then the inequality holds. Each firm can then increase profits by increasing the distance between

the prices. The mixed-strategy equilibrium, where they set one price for all units, no longer holds. If, however $q^a = .7$, $q^b = .1$, $q^c = .1$ and $q^d = .1$ then the equilibrium continues to hold. For any distribution that is not skewed in this way the equilibrium holds. QED. ∎

Proposition 6 *Under a uniform-price rule there is no flat-supply equilibrium when demand is uniformly distributed on the interval $[0, 2k]$.*

Proof. We can construct an equilibrium where the firms mix between flat supply functions as in the discriminatory case. Suppose firm 2 is submitting a price $p \in (v^U, \overline{p})$ for all units, according to the distribution function $F(p)$. Let $f(p)$ be the corresponding density function. Then player 1's expected payoff from submitting a price p_1 for all units is

$$
\Pi(p_1) = q^h \left(\int_{p_1}^{\overline{p}} (p-c)kf(p)dp + \int_{v^U}^{p_1} (p_1-c)(d^h - k)f(p)dp \right)
$$
$$
+ q^l \int_{p_1}^{\overline{p}} (p_1-c)d^l f(p)dp
$$

where $q^h = \text{Prob}(d > k)$, $d^h = E(d \mid d > k)$, $q^l = \text{Prob}(d < k)$, $d^l = E(d \mid d < k)$. In equilibrium $\Pi'(p_1) = 0$ for all $p_1 \in (v^U, \overline{p})$. This gives

$$
A - Bf(p)(p-c) - CF(p) = 0
$$

where $A = q^l d^l$, $B = (2k - d^h)q^h + q^l d^l$, $C = q^l d^l - (d^h - k)q^h$

The unique solution of this differential equation with boundary condition $F(\overline{p}) = 1$ when $C \neq 0$ is

$$
f(p) = \frac{A-C}{B} \frac{(\overline{p}-c)^{\frac{C}{B}}}{(p-c)^{\frac{C}{B}-1}} \tag{9}
$$

$$
F(p) = \frac{A}{C} - \frac{(A-C)}{C} \left(\frac{\overline{p}-c}{p-c} \right)^{\frac{C}{B}} \tag{10}
$$

The unique solution when $C = 0$ is

$$
f(p) = \frac{A}{B} \frac{1}{(p-c)} \tag{11}
$$

$$
F(p) = 1 - \frac{A}{B} Log \left(\frac{\overline{p}-c}{p-c} \right) \tag{12}
$$

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However, this is not in general an equilibrium in the multi-unit case. Using the previous notation , if firm 2 sets an increasing supply function the profit function (focussing on terms that include p_n) is

$$
\begin{aligned}
\Pi(P) \;=\; & K(p_1,....p_{n-1}) \\
& +q^b\left(\begin{array}{c} \int_{p_n}^{\overline{p}}(p_n-c)d^b f(p)dp \\ +\int_{p_{n-1}}^{p_n}(p-c)(k-\alpha)f(p)dp \end{array}\right) \\
& q^c\left(\begin{array}{c} \int_{p_n}^{\overline{p}}(p-c)k f(p)dp \\ +\int_{p_{n-1}}^{p_n}(p-c)(k-\alpha)f(p)dp) \end{array}\right) \\
& +q^d(\int_{p_n}^{\overline{p}}(p-c)k f(p)dp + \int_{v^U}^{p_n}(p_n-c)(d^d-k)f(p)dp)
\end{aligned}
$$

Unlike in the discriminatory case, it is not generally optimal to set $p_n = p_{n-1}$. The first derivative with respect to p_n is

$$
\begin{aligned}
\frac{\partial \Pi(P)}{\partial p_n} \;=\; & q^b d^b - F(p_n)(q^b d^b - q^d(d^d-k)) \qquad\qquad (13) \\
& -f(p_n)(p_n-c)(q^b(d^b-(k-\alpha)+\alpha q^c + q^d(2k-d^d))
\end{aligned}
$$

Consider the case where $d \sim U[0,2k]$. Then $q^a = (k-\alpha)/2k$, $q^b = a/2k$, $q^c = (k-a)/2k$, $q^d = a/2k$, $d^a = (k-a)/2$, $d^b = (2k-a)/2$, $d^c = (3k-a)/2$ and $d^d = (4k-a)/2$. Substituting for $f(p_n)$ and $F(p_n)$ using (11) and (12) the first derivative is

$$
\frac{1}{4}\frac{\alpha(k-\alpha)}{k}
$$

which is always positive in which case each firm can gain by differentiating prices. ■

Proposition 7 *Under a uniform-price rule there is an independent increasing-supply equilibrium when demand is uniformly distributed on the interval $[0,2k]$ which leads to an average price that is greater than in the flat-supply equilibrium.*

Proof. *Let $q^x = Prob(x-1 < d < x)$ and $d^x = E(d \mid x-1 < d < x)$. Assume firm 2 is choosing its highest price according to some density $f_k(p)$ on some interval $[v_k,\overline{p}]$, the second highest according to some density $f_{k-1}(p)$ on some interval $[v_{k-1},v_k]$.... and the nth price according to some density*

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$f_n(p)$ *on some interval* $[v_n, v_{n+1}]$. *The profit function of firm 1 if it submits a price vector* P *with a price in each of the intervals is,*

$$
\begin{aligned}
\Pi(p) \;=\;& q^1 \left(\int_{p_1}^{v_2} d^1 (p_1 - c) f_1(p) dp \right) \\
&+ q^2 \left(\begin{array}{c} \int_{p_1}^{v_2} (p - c) f_1(p) dp \\ + \int_{v_1}^{p_1} (d^2 - 1)(p_1 - c) f_1(p) dp \end{array} \right) \\
&\quad \cdot \\
&\quad \cdot \\
&+ q^{2x-1} \left(\begin{array}{c} \int_{p_x}^{v_{x+1}} (d^{2x-1} - (x-1))(p_x - c) f_x(p) dp \\ + \int_{v_x}^{p_x} (x-1)(p - c) f_x(p) dp \end{array} \right) \\
&+ q^{2x} \left(\begin{array}{c} \int_{p_x}^{v_{x+1}} x(p - c) f_x(p) dp \\ + \int_{v_x}^{p_x} (d^{2x} - x)(p_x - c) f_x(p) dp \end{array} \right) \\
&\quad \cdot \\
&\quad \cdot
\end{aligned}
$$

Now for this to be an equilibrium, firm 1 must be indifferent between any price in each interval. We therefore have k first order conditions that give rise to k differential equations. The xth first order condition is

$$
\begin{aligned}
\frac{\partial \Pi(P)}{\partial p_x} \;=\;& q^{2x-1} \left(d^{2x-1} - (x-1) \right) \quad\quad\quad\quad (14) \\
&- (p_x - c) f_x(p_x) \left(\begin{array}{c} q^{2x} (2x - d^{2x}) \\ + q^{2x-1} (d^{2x-1} - 2(x-1)) \end{array} \right) \\
&- F_x(p_x) \left(\begin{array}{c} q^{2x-1} (d^{2x-1} - (x-1)) \\ - q^{2x} (d^{2x} - x) \end{array} \right)
\end{aligned}
$$

The solution is of the type given in the proof of proposition 6. Finally, to confirm that this is an equilibrium we must check that it is globally optimal to set prices in these intervals. Suppose instead that player 1 submits $x > 1$ prices in the interval $[v_m, v_l]$ with y prices below v_m and $k - y - x$ prices above v_l. In total, there will be $y + m - 1$ prices below v_m, $x + 1$ prices in the interval $[v_m, v_l]$ and $2k - y - m - x$ prices above v_l. Let q_m be the price firm 2 sets on the interval $[v_m, v_l]$. Now consider a unit firm 1 has bid at p_t on the interval $[v_m, v_l]$. This unit will be ranked as the $m + t$ lowest when $q_m < p_t$ and as the $m + t - 1$ lowest when $q_m > p_t$. Hence the terms in the profit function that

include p_t are

$$q^{m+t-1} \left(\begin{array}{c} \int_{p_t}^{v_l} \left(d^{m+t-1} - (m-1)\right) (p_t - c) f_m(q_m) dq_m \\ + \int_{\max[v_m, p_{t-1}]}^{p_t} (t-1)(q_m - c) f_m(q_m) dq_m \end{array} \right)$$

$$+ q^{m+t} \left(\begin{array}{c} \int_{p_t}^{\min[v_l, p_{t+1}]} t(q_m - c) f_m(q_m) dq_m \\ + \int_{v_m}^{p_t} \left(d^{m+t} - m\right) (p_t - c) f_m(q_m) dq_m \end{array} \right)$$

The first derivative of firm 1's profit with respect to p_t is therefore

$$\frac{\partial \Pi(p)}{\partial p_t} = q^{m+t-1} \left(d^{m+t-1} - (m-1) \right) \tag{15}$$

$$- F_m(p_t) \left(\begin{array}{c} q^{m+t-1} \left(d^{m+t-1} - (m-1) \right) \\ -q^{m+t} \left(d^{m+t} - m \right) \end{array} \right)$$

$$- (p_t - c) f_m(p_t) \left(\begin{array}{c} q^{m+t-1} \left(d^{m+t-1} - (m+t-2) \right) \\ +q^{m+t} \left(m+t - d^{m+t} \right) \end{array} \right)$$

For a uniform distribution, $q^x = \frac{1}{2k}$ and $d^x = \frac{2x-1}{2}$. Substituting these values into (11) (using values from (14)) and then substituting into (15) gives,

$$\frac{\partial \Pi(p)}{\partial p_t} = \frac{1}{2k}(t - m) \tag{16}$$

Firm 1 can gain by increasing p_{y+x} to v_l if $y + x > m$ and by reducing p_{y+1} to v_m if $y + 1 < m$. If $x > 1$ then at least one of these inequalities hold. It is therefore not optimal to submit two or more prices on one interval. Each firm can get the flat-supply equilibrium payoff by submitting all units at $\overline{p}$. However, (16) shows that the firm would strictly increase profit by reducing the price of a unit to the second highest interval. Hence the average cost to the buyer is strictly greater than in the flat-supply equilibrium. ∎

References

[1] Ausubel, L.M. and P. Cramton. 1995. Demand Reduction and Inefficiency in Multi-Unit Auctions. Working Paper 96-07. University of Maryland.

[2] Back, K. and J. Zender. 1993. Auctions of Divisible Goods: On the rationale for the Treasury Experiments. *Review of Financial Studies 6*, 733-764.

[3] Binmore, K. and J. Swierzbinski. 2002. Treasury Auctions, Uniform or Discriminatory? Forthcoming in the *Journal of Economic Design.*

[4] Borenstein, S, J. Bushnell and F. Wolak. 2000. Diagnosing Market Power in California's Restructured Wholesale Electricity Market. Working paper, Stanford University.

[5] Engelbrecht-Wiggans, R. and C.M. Kahn. 1998. Multi-Unit Auctions with Uniform Prices. *Economic Theory 12,* 227-258.

[6] Engelbrecht-Wiggans, R. and C.M. Kahn. 1998. Multi-Unit Pay-Your Bid Auctions with Variable Awards. *Games and Economic Behavior* 23, 25-42.

[7] Fabra, N, N-H von der Fehr, and D. Harbord. 2002. Designing Electricity Auctions. Forthcoming in the *Rand Journal of Economics.*

[8] Fehr, N-H von der, and D. Harbord. 1993. Spot Market Competition in the UK Electricity Industry. *Economic Journal* 103, 531-546.

[9] Fehr, N-H von der, and D. Harbord. 1998. Competition in Electricity Spot Markets: Economic Theory and International Experience. Memorandum No.5/1998, Department of Economics, University of Oslo.

[10] Green, R.J. and D.M. Newbery. 1992. Competition in the British Electricity Spot Market. *Journal of Political Economy* 100, 929-953.

[11] Klemperer, P.D. and M.A. Meyer. 1989. Supply Function Equilibria in Oligopoly Under Uncertainty. *Econometrica* 57,1243-1277.

[12] Noussair, C. 1995. Equilibria in a Multi-Object Uniform Price Sealed Bid Auction with Multi-Unit Demands. *Economic Theory* 5, 337-351.

[13] Ofgem (1999). *The New Electricity Trading Arrangements: Ofgem/DTI Conclusion Document* October 1999.

[14] Wilson, R. 1979. Auctions of Shares. *Quarterly Journal of Economics* 93, 675-698.

[15] Wolak, F. and R. Patrick. 1996. The Impact of Market Rules and Market Structure on the Price Determination Process in the England and Wales Electricity Market. CEPR Discussion Paper 463, Stanford University.