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Liquidity and Asset Prices

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Liquidity and Asset Prices

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I. Introduction.

People in financial markets pay considerable attention to the behavior of central bank. It is often said that there is a shortage of liquidity and that the central bank supplies liquidity to the market. Yet standard asset pricing models seem to have little scope for systematic analysis of the effect of monetary policy on liquidity and asset prices. Some well-known puzzles in the asset pricing literature, such as the low risk-free rate puzzle and the equity premium puzzle, seem to presume that the underlying economy is non-monetary. But such puzzles can be related to a traditional question in monetary economics: why do people hold monetary assets, even though the rate of return is low and often dominated by the return of some other assets?[1] This paper develops a canonical model of a monetary economy, in order to examine the interaction between circulation of monetary assets, monetary policy, resource allocation, and asset prices.

We broadly define monetary assets, or liquid assets, as the assets that can be readily sold in the market and can be held by a number of people in succession before maturity. When some asset circulates among many people as a means of short-term saving (liquidity), it also serves as a medium of exchange (money), because people hold it not for their own consumption at its maturity date but for future exchange. Thus we will use liquid asset and monetary asset interchangeably. We first ask in what environments is the circulation of a monetary asset essential for the smooth running of the economy. Perhaps the closest ancestor of this paper is [Townsend(1987)]. While Townsend contrasts cash and real assets, we contrast broad liquid assets and illiquid assets. That is, in our paper, all the assets are real; a broad liquid asset is not denominated by cash, and we ignore the issue of fiat money and inflation.[2]

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[1] We explore the exact connection between these puzzles by developing a formal model. Although the robustness of the equity premium puzzle is debatable empirically, the puzzle of monetary assets appears to be robust. See [Mehra and Prescott(1985)], [Weil(1989)] and [Campbell, Lo and Mackinlay(1997)].

[2] We do not consider some popular monetary frameworks, the cash-in-advance model or the dynamic sticky-price model, as models of a monetary economy,

In order to analyze the role of monetary assets for resource allocation, we consider an economy in which output is produced from two types of asset, capital and land. Capital stock can be accumulated through productive investment, while the supply of land is fixed. We depart from a standard model of a stochastic production economy with a representative agent (a real business cycle model) in two aspects. First, we assume that only a fraction of agents has a productive investment opportunity to accumulate capital stock at each point in time, even though agents are equally likely to find investment opportunities in the future. We also assume that there is no insurance contingent on the arrival of investment opportunity. Thus, the economy must transfer purchasing power through financial markets from those who do not have a productive investment opportunity to those who have.[3] The second departure is that, at the time of productive investment, people can sell only a fraction of their capital stock (or equivalently, a claim to the future returns from capital stock). Thus capital stock is an asset with

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because in those frameworks the circulation of monetary assets is not indispensable for efficient resource allocation (on the contrary, it often distorts an otherwise efficient allocation). Some other monetary frameworks, the over-lapping generations models and the random matching models, do explain why the circulation of money improves efficiency. These models, however, are not very easy to apply to an economy with well-developed financial markets.

[3] A large part of the asset pricing literature with credit constraints uses an endowment economy, in which the focus is on risk-sharing for households who face idiosyncratic utility shocks or income shocks. (See [Cochrane(2001)].) Here, we consider a production economy in which the role of financial markets is to transfer resources to those agents who have productive investment opportunity. [Holmströn and Tirole(2000)] develops a liquidity-based asset pricing model of a three-period production economy with financial intermediaries in which the arrival of an investment opportunity is contractible. Our analysis largely abstracts from financial intermediation and contingent contracting of this kind in order to concentrate on the dynamic general equilibrium effects. Our framework is perhaps more comparable to a standard asset pricing model, given that in our economy, agents are identical ex ante, risk-averse and infinitely lived.

limited liquidity. Investing people may therefore face binding credit constraint. One interpretation of our model is that the productive investment opportunities disappear so quickly that investing agents do not have enough time to raise funds against their entire capital holding, nor to process an insurance claim, in order to finance new investment. In contrast, land is a liquid asset, and people can raise fund against the entire land holdings at the time of investment. (In reality, of course, land is often less liquid than capital. The term 'land' in this paper may be taken to represent the productive assets of old and well-established sectors of the economy. Such sectors consist of publicly-traded firms; their stock market is well-organized and their productive assets are relatively constant. In contrast, the term 'capital' might represent the productive assets of new and dynamic sectors of the economy, comprising less-established businesses.)

We show that the circulation of the liquid asset is essential for resource allocation, -- i.e. the economy is 'monetary', -- if, each agent rarely has a productive investment opportunity; investing agents can sell only a small fraction of capital; and the income share of land is small relative to capital. In the monetary economy, people with investment opportunities are credit constrained. Also, there is liquidity premium (the gap in the expected rates of returns between the illiquid asset, capital, and the liquid asset, land). And the expected rates of return on the liquid asset is lower than time preference rate. These phenomena are closely related. If people anticipate a binding credit constraint at the time of investment, they will hold the liquid asset in their portfolios even if its expected rate of return is dominated by that of the illiquid asset, and even if it is lower than their time preference rate, because the liquid asset is more valuable for financing downpayment for investment than is the illiquid asset. That is, the liquidity premium, the low liquid asset return, and the credit constraint for investing agents are all features of the monetary economy.

In the later part of the paper, we extend the basic model in two directions. First, we introduce labor as a factor of production and workers as a new group of agents. Workers do not have productive investment opportunities, nor can they borrow against their future wage income. We show that, when the economy is monetary, the workers do not save, while entrepreneurs, who anticipate the arrival of an investment opportunity in the future, save with both the liquid asset and the illiquid asset. The saving behavior of workers and entrepreneurs are different because they have different expectations of productive investment opportunity. It is not because they have different preferences. The second extension is to introduce government. We consider the government as a large agent who does not have any productive investment opportunities. The government owns a fraction of capital stock and land, taxes workers, and purchases goods. The government behavior is taken to be exogenous. We examine how the portfolio policy and the fiscal policy of the government affect aggregate resource allocation and asset prices. We show that an open market operation to buy the illiquid asset and to sell the liquid asset, has expansionary effects on aggregate output even in the long run, because this operation increases the liquidity of the private sector, which facilitates the transfer of more resources from saving agents to investing agents. Again, the difference in saving behavior between workers and entrepreneurs, and the expansionary effect of the open market operation are both closely related to the other features of the monetary economy, such as the binding credit constraint faced by the investing entrepreneurs, the liquidity premium, and the low rate of return on the liquid asset. Because of the low rates of return on assets relative to the time preference rate, workers do not save. Entrepreneurs, anticipating the binding credit constraints, save in assets, despite the low rates of return on assets, in order to finance the downpayment for future investment. The government's open market operation increases the ratio of the liquid asset to the illiquid asset of the private sector. Such a policy stimulates aggregate output, because it is the private holding of the liquid asset in particular that lubricates the transfer of resources in the monetary economy.

II. Basic Model

Consider a continuous-time economy with one homogeneous output. There is a continuum of infinitely-lived agents with population size of unity. The utility of an agent at date 0 is described as:

$$(1) \qquad V_0 \quad = \quad E_0 \left[\int_0^{\infty} \ln c_t \, e^{-rt} \, dt \right],$$

where c_t is consumption rate at date t, r is a constant time preference rate, ln x is natural log of x, and $E_t[x]$ is expected value of x conditional on information at date t.

The individual agent produces output from two types of homogeneous assets, capital stock and land, according to a constant returns to scale production function:

$$(2) \qquad y'_t \quad = \quad a_t \, k_t'^{\alpha} \, \ell_t'^{1-\alpha}, \qquad\qquad 0 < \alpha < 1.$$

where y'_t is output, k'_t is capital stock and ℓ'_t is land used for production. The variable a_t is aggregate productivity, which is common to all individuals. We assume that the aggregate productivity follows a two-points Markov process:

$$(3) \qquad a_t \quad = \quad a_h, \text{ or } a_\ell, \text{ where } a_h \geq a_\ell, \text{ and}$$

that the arrival rate of switch of the productivity is constant and equal to η. The individual agents own capital and land, but can freely rent capital and land for production at perfectly competitive rental markets. Capital stock depreciates at constant rate δ. Total supply of land is fixed and normalized to be unity.

Each agent meets an opportunity to invest on capital according to a Poisson process of arrival rate π. When an agent has the productive investment opportunity, the agent can convert homogeneous goods into capital one-for-one instantaneously. In order to finance this investment, the agent can sell the entire land holding, but the agent can sell only a fraction θ of his capital held before the investment, and also can sell a fraction θ of the new capital just invested. We consider θ as an exogenous parameter which represents the limited liquidity of existing capital and the newly-installed capital. An alternative interpretation of θ is a particular shape of transaction cost of capital, in which the cost of selling the first θ fraction is zero and the cost for the rest is infinite at the time of investment.[4] On the other hand, land (or a claim to the future returns on land) is a liquid asset, which serves as money in our model. The value of θ also reflects how quickly the agent must exploit the investment opportunity once he finds such opportunity. Here, we assume that the opportunity disappears very quickly after the agent finds it, and that the agent does not have much time to sell the entire capital, even if the agent can sell all the capital eventually. We also assume that the arrival of the investment opportunity is not contractible so that the agent cannot arrange an insurance contract contingent on the arrival of the investment opportunity. (Again, we consider the environment in which the investment opportunity is gone by the time the agent may receive payment from the insurance company with verification of the opportunity.)[5] We also assume that the individual agent cannot short-sale capital nor land. (This does not mean the agent cannot borrow. On the contrary, we allow the agent to sell the entire land and θ fraction of capital, even if the agent is employing land and capital for production. This means that the agent can

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[4] In the earlier paper [Kiyotaki-Moore(2000)], we explain why the producer cannot borrow against more than a fraction of future output and why those who lend to this producer cannot easily sell the producer's paper before maturity. Here, we assume that the producer can eventually sell all the ownership of the capital employed for his own production, and that there is no difference in liquidity between the capital used for his own production and capital employed by the other producers because there is a perfectly competitive rental market of capital. Also, because the production function is constant returns to scale, the profit is zero in the equilibrium and the question of who produce becomes less important.

[5] [Hölmstrom-Tirole(2000)] style insurance arrangement is not incentive compatible here, because everyone can equally participate in financial markets.

borrow up to 100% of land and 100xθ % of capital.[6])

Let P_t be the price of land, let q_t be price of capital installed in terms of goods. The flow-of-fund constraint of the investing agent is:

$$(4) \qquad (1 - \theta q_t)\, i_t \quad \le \quad P_t \ell_t + \theta q_t k_t,$$

where i_t is investment, ℓ_t is land and k_t is capital owned before the investment. The left-hand side is the required downpayment for investment, i.e., the investment expenditure which is not financed by sales of θ fraction of capital just installed. The right-hand-side is maximum sales of land and capital available for paying the downpayment.

Let b_t be the total value of land (ℓ_t) and capital (k_t) owned by an individual agent:

$$(5) \qquad b_t \quad = \quad P_t \ell_t + q_t k_t.$$

If the agent has an productive investment opportunity at date t, the agent can choose whether to invest the maximum amount subject to the flow-of-funds constraint (4) or not. After the maximum investment, the agent has sold all the land and holds 1-θ fraction of capital previously held and capital newly invested. Thus the total asset after the investment would be:

$$(6) \qquad b_{t+0} \quad = \quad \mathrm{Max}\left(q_t (1-\theta)(k_t + i_t), \quad b_t \right)$$

$$= \quad \mathrm{Max}\left(\frac{(1-\theta) q_t}{1 - \theta q_t} (P_t \ell_t + k_t), \quad b_t \right).$$

Then, the agent will invest up to the maximum if $q_t > 1$, and will not invest if $q_t < 1$, and indifferent if $q_t = 1$. The value of q_t is equal to so-called Tobin's q_t, the ratio of the value of installed capital and production cost of capital. The investment choice is similar to Tobin's q theory of investment.

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[6] Here, however, the repayment amount of the borrower is fully indexed by the price of the collateral assets. Therefore, there is no problem of the leverage or the debt-overhang studied in [Kiyotaki-Moore(1997)].

Tobin's q may exceeds unity here, because only a small fraction of agents have the opportunity to invest on capital stock at each point in time and they are credit constrained. In (6), we can think of $(1-\theta)q_t/(1-\theta q_t)$ as the rate of return on funds for the investing agent, because the denominator is the required downpayment and the numerator is the return from unit investment with selling θ fraction of capital invested. The investing agent behaves as if he values capital in replacement cost k_t instead of market value $q_t k_t$ in financing the downpayment for investment, while he always values liquid land at market value $P_t \ell_t$. Let R_t^K and R_t^L be rental income of capital and land per unit. The accumulation of total asset of the agent between date t and t+dt will be:

$$(7) \quad db_t = (R_t^L dt + dP_t)\ell_t + (R_t^K dt + dq_t)k_t - c_t dt$$

$$+ \mathrm{Max}\left(\frac{(1-\theta)q_t}{1-\theta q_t}(P_t\ell_t + k_t) - b_t,\ 0\right) dM_t,$$

where $dM_t = 1$ if the agent has an investment opportunity at date t and $dM_t = 0$ otherwise.

The aggregate state of the economy is summarized by the aggregate capital stock and the aggregate productivity; $s_t = (K_t, a_t)$. The competitive equilibrium is described as price functions $P_t = P(s_t)$, $q_t = q(s_t)$, $R_t^K = R^K(s_t)$ and $R_t^L = R^L(s_t)$ and quantities of consumption, investment, capital stock and output, such that (i) The individual agent chooses rules of consumption, investment and asset portfolio $\{c(b_t,s_t),\ i(b_t,s_t),\ k(b_t,s_t),\ \ell(b_t,s_t)\}$ and production to maximize the utility (1) subject to the flow-of-funds constraints (4),(7) and the balance sheet constraint (5), taking the price functions as given; (ii) sum of the individual capital holding is equal to the aggregate capital stock; (iii) sum of the individual land holding is equal to total land supply $L_t = 1$; (iv) aggregate consumption and investment is equal to aggregate output, and (v) rental markets for capital and land clear.

In the following, we concentrate on the case in which Tobin's q exceeds unity so that the credit constraint is binding for the investing agents, and we will later derive the condition that guarantees $q_t > 1$. In order to characterize the individual agent's utility maximization, let V(b,s) be the value function of individual with total asset b when the aggregate state is s. Then Bellman equation would be:

$$\text{(8)}\quad r\,V(b,s) = \underset{c,\ell,k}{\text{Max}} \left\{ \ln c + V_b(b,s)\left[(R^L+\dot{P})\ell + (R^K+\dot{q})k - c\right]_{da=0} \right.$$

$$+ \pi \left[V\left(\frac{(1-\theta)q}{1-\theta q}(P\ell+k), s\right) - V(b,s)\right]$$

$$\left. + \eta \left[V(P(s')\ell+q(s')k, s') - V(b,s)\right]_{da\neq 0} \right\}, \quad \text{subject to (5)},$$

where s is the aggregate state at date t and s' is the state at date t+dt. The first line in the right-hand side is the utility of consumption and the value of saving when there is no switch in aggregate productivity. With the same productivity, there is no discontinues jump in the asset prices and $\dot{x}$ denotes $\frac{dx}{dt}$. The second line is the expected gains in the value with the arrival of productive investment opportunity, corresponding to equation (6). The last term is the capital gains associated with the switch of the aggregate productivity. The first order conditions for consumption and portfolio are:

$$\text{(9)}\qquad \frac{1}{c} = V_b(b,s),$$

$$\text{(10)}\qquad V_b(b,s)\left[\frac{R^L+\dot{P}}{P}\right]_{da_t=0} + \eta\, V_b(P(s')\ell+q(s')k, s')\left[\frac{P(s')}{P}\right]_{da_t\neq 0}$$

$$+ \pi\, V_b\left(\frac{(1-\theta)q}{1-\theta q}(P\ell+k), s\right) \frac{(1-\theta)q}{1-\theta q} = \pi\, V_b\left(\frac{(1-\theta)q}{1-\theta q}(P\ell+k), s\right) \frac{(1-\theta)q}{1-\theta q}\,\frac{1}{q}$$

$$+ V_b(b,s)\left[\frac{R^K+\dot{q}}{q}\right]_{da_t=0} + \eta\, V_b(P(s')\ell+q(s')k, s')\left[\frac{q(s')}{q}\right]_{da_t\neq 0}$$

Equation (9) means the marginal utility of consumption must be equal to the marginal value of asset for the optimal consumption choice. Equation (10) implies that the expected rate of returns on liquid land and illiquid capital should be equal in terms of utility for the optimal portfolio. In particular, the second line is the expected return for financing downpayment of investment, in which the rate of return on illiquid capital is $(1/q_t)$ times as much as the rate of return on liquid land. Then, we have:

$$(11) \qquad V(b_t, s_t) = v(s_t) + \frac{1}{r} \ln b_t,$$

$$(12) \qquad c_t = r\, b_t,$$

$$(13) \qquad \left[\frac{R_t^L}{P_t} - \frac{R_t^K}{q_t} \right] + \pi \frac{q_t - 1}{q_t} \frac{P_t \ell_t + q_t k_t}{P_t \ell_t + k_t}$$

$$+ \left[\frac{\dot{P}_t}{P_t} - \frac{\dot{q}_t}{q_t} \right]_{da_t = 0} + \eta \left[\left(\frac{P_{t+dt}}{P_t} - \frac{q_{t+dt}}{q_t} \right) \frac{P_t \ell_t + q_t k_t}{P_{t+dt} \ell_t + q_{t+dt} k_t} \right]_{da_t \neq 0} = 0.$$

Because the instantaneous utility function is log of consumption rate, the value function is log linear function of total asset in equation (11). Equation (12) implies consumption rate is equal to the time preference rate times total asset. Equation (13) is the optimal portfolio condition, roughly implying that the expected rate of returns on land and capital should be equal in terms of goods, after adjusting for the difference in liquidity and risk. The first term is the difference in rental income rate between land and capital. The second term is the expected liquidity advantage of land over capital, which is proportional to the gap between Tobin's q and 1. The third term is the difference in capital gains rate on land and capital when there is no change in productivity so that the change in the asset prices are predictable over time. The last term is the effect of capital gains associated with a discrete change in productivity, taking account of risk aversion. Because the fraction of agents who face credit constraints is infinitesimal, almost everyone will choose the same ratio of capital to land which satisfies (13).[7] For some readers, the discrete-time model may be more

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[7] Immediately after productive investment, the agent holds only capital stock and no land. Here, because the population of agents who invest in

familiar than the continuous-time model. In Appendix, we will describe the discrete-time formulation of the individual maximization, and then derive the above equations as a limit when the length of period goes to infinitesimal.

When Tobin's q exceeds unity, the flow-of-funds constraint (4) holds with equality for all the investing agents. Since the arrival rate of productive investment opportunity is independent across agents, we can aggregate the individual flow-of-funds constraints as:

$$(14) \qquad (1 - \theta q_t)\, I_t \quad = \quad \pi\, (P_t + \theta q_t K_t),$$

where I_t is the aggregate investment, noting that the aggregate land is unity. When investing agents face credit constraint, the aggregate investment is an increasing function of capital price and land price. Aggregate capital stock evolves over time as:

$$(15) \qquad \dot{K}_t \quad = \quad I_t - \delta K_t.$$

Through competitive rental market, both the marginal product of capital and the marginal product of land are equalized across producers. Thus the aggregate output Y_t becomes a function of aggregate capital stock as:

$$(16) \qquad Y_t \quad = \quad a_t\, K_t^{\alpha}\ .$$

Aggregating the individual's consumption and investment, we have the goods market clearing condition as:

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infinitesimal time interval of [t, t+dt] is equal to πdt, and because it takes infinitesimal time to adjust the portfolio after finishing the investment, the proportion of agents who have different assets ratio from the majority of agents is infinitesimal. In a discrete-time model, we would need to take accounts of these complications more explicitly, which we want to avoid here by using a continuous-time model.

(17) $$Y_t = I_t + r(P_t + q_t K_t).$$

In the competitive rental markets, the rental income of capital and land per unit are equal to the marginal products of capital and land after taking accounts the depreciation:

(18a) $$R_t^K = \alpha\, a_t\, K_t^{\alpha-1} - \delta q_t,$$

(18b) $$R_t^L = (1-\alpha)\, a_t\, K_t^{\alpha}.$$

Since almost everyone has the same land-capital ratio, we can aggregate the optimal portfolio condition (13) to derive the assets market equilibrium condition as:

(19) $$\left[\frac{1-\alpha}{P_t} - \frac{\alpha}{q_t K_t}\right] a_t K_t^{\alpha} + \delta + \pi\, \frac{q_t - 1}{q_t}\, \frac{P_t + q_t K_t}{P_t + K_t}$$

$$+ \left[\frac{\dot{P}_t}{P_t} - \frac{\dot{q}_t}{q_t}\right]_{da_t=0} + \eta \left[\left(\frac{P_{t+dt}}{P_t} - \frac{q_{t+dt}}{q_t}\right) \frac{P_t + q_t K_t}{P_{t+dt} + q_{t+dt} K_t}\right]_{da_t \neq 0} = 0.$$

The competitive equilibrium is described as $P(K_t, a_t)$, $q(K_t, a_t)$, $I(K_t, a_t)$ and stochastic process of K_t and a_t, which satisfy (3),(14),(15),(16),(17) and (19).

Our economy is characterized fully by the goods market clearing (17), the assets market clearing (19), the flow-of-funds constraint of the investing agents (14), and the technological constraints on production (16),(17) and (3). Our model is as simple as standard asset pricing models of production economy (such as [Merton(1975)] and [Brock(1982)]), real business cycle models, and IS-LM models. The main difference from standard asset pricing models is the expected advantage of the liquid asset over illiquid capital in financing investment in (19), which in turn depends upon Tobin's q (a measure of tightness of the credit constraint of the investing agents) Our model differs from standard real business cycle models (such as [Kydland and

Prescott(1982)], because we have two-way interaction between the asset prices and the aggregate quantities. A typical real business cycles model can determine the quantities first, before deriving the implied prices. In this aspect, our model is closer to traditional IS-LM model. Our framework, however, differs substantively beyond the modeling strategy, because, while IS-LM compares cash and interest-bearing assets in determining the nominal interest rate, we contrast between the broad liquid asset and the illiquid asset in determining the liquidity premium.[8]

Before analyzing the dynamics, let us examine the steady state equilibrium for a constant aggregate productivity, i.e., $a_h = a_\ell = a$. In the steady state, capital stock is constant so that $I = \delta K$, and

$$\text{(20a)} \qquad (1 - \theta q)\delta K = \pi(P + \theta qK),$$

$$\text{(20b)} \qquad aK^{\alpha} = \delta K + r(P + qK),$$

$$\text{(20c)} \qquad \left[\frac{1-\alpha}{P} - \frac{\alpha}{qK}\right]aK^{\alpha} + \delta + \pi\,\frac{q-1}{q}\,\frac{P + qK}{P + K} = 0.$$

From these conditions of the steady state equilibrium, we can show the following proposition:

<u>Proposition 1:</u> Suppose that the parameters of the economy satisfies

$$\text{(Assumption 1)} \qquad \theta < \theta^{*} \equiv \frac{1}{\pi+\delta}\left(\delta - \pi\,\frac{1-\alpha}{r}\,\frac{r+\delta}{\alpha}\right).$$

Then, in the steady state equilibrium:

(i) Tobin's q exceeds 1, so that investing agents are credit constrained;

$$\text{(ii)} \qquad \frac{R^L}{P} < \frac{R^K}{q} < r;$$

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[8] In this respect, we are close to [Keynes(1936)], which defines 'money' as broad liquid assets, including treasury bills.

(iii) $K < K^*$, where K^* is the capital stock at the first-best steady state, which solves:

$$\alpha \, a \, (K^*)^{\alpha-1} = r + \delta.$$

(Proof in Appendix)

Assumption 1 implies that the fraction of capital which the agents can sell in order to finance the investment is smaller than the critical level θ^*. The value of θ^* is a decreasing function of the arrival rate of investment opportunity π, and the ratio of land value to capital under the first-best allocation: $\frac{(1-\alpha)Y/r}{\alpha Y/(r+\delta)}$. Thus, Proposition 1(i) means that investing agents are credit constrained, if and only if capital is sufficiently illiquid in relation to the arrival rate of investment opportunity and the land value - capital ratio. Proposition 1(ii) implies that the rate of return on the liquid asset, land, is dominated by the rate of return on the illiquid asset, capital, which in turn is smaller than the time preference rate. The gap between the rates of returns on illiquid and liquid assets may be called as 'liquidity premium,' and the liquidity premium arises if and only if the investing agents are credit constrained. From (20c), the magnitude of the liquidity premium is roughly equal to $\pi(q-1)$, which can be substantial. For an example, if Tobin's q is equal 5% above 1 and each agent has investment opportunity once every two years on average, then the liquidity premium is 2.5% annual rate.[9] Proposition 1(iii) says that there is under-investment relative to the first-best allocation, if the credit constraint is binding so that not enough resources is transferred from saving agents to investing agents. All of these features, credit constraint, liquidity premium, low liquid asset return and under-investment are unique features of "monetary economy," in which circulation of the liquid asset is essential for the smooth running of the economy.

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[9] If the agents were heterogeneous in technology, then the arbitrage equation (19) holds only for those who hold both liquid and illiquid assets. On the other hand, if some people's Tobin's q is 0.05% above 1 and they receive investment opportunity 50 times a year on average, then they hold both assets with liquidity premium of 2.5%.

The reverse of Proposition 1 is also true. Suppose that $\theta \geq \theta^*$. Then, the steady state equilibrium achieves the first-best allocation; Tobin's q is equal to 1 and the investing agents are not credit constraint; and the rates of returns on capital and land are both equal to the time preference rate. Notice that, if $\theta > \delta/(\pi+\delta)$, then the economy achieves the first best allocation, even if liquid land is unimportant for production, $\alpha \cong 1$.[10] Under such environment, the economy ceases to be monetary, in the sense that the circulation of liquid monetary asset is no longer necessary for the efficient allocation.

We now examine the dynamics of the economy with stochastic fluctuations of the aggregate productivity. Here, we concentrate on local dynamics around the steady state against a small fluctuation of aggregate productivity, i.e.,

(21) $a_h = a(1+\Delta)$, and $a_\ell = a(1-\Delta)$, where Δ is infinitesimally positive.

We also assume that Assumption 1 holds with strict inequality so that the investing agents are always credit constrained in the neighborhood of the steady state equilibrium. Because it is easier to analyze the model in intensive form, let us define $i_t = I_t/K_t$ (investment rate), $p_t = P_t/K_t$ (land value - capital ratio) and $y_t = Y_t/K_t$ (output-capital ratio). Equations (14), (15), (16), (17) and (19) are now:

(22) $$(1 - \theta q_t)\, i_t = \pi(p_t + \theta q_t),$$

(23) $$\dot{K}_t/K_t = i_t - \delta,$$

(24) $$y_t = a_t K_t^{\alpha-1} = i_t + r(p_t + q_t).$$

10 In [Kiyotaki-Moore(2000)] shows similar propositions (Propositions 1 and 2) in a slightly different environment.

$$(25)\quad \left[\frac{1-\alpha}{p_t} - \frac{\alpha}{q_t}\right] y_t + i_t + \pi \frac{q_t-1}{q_t}\frac{p_t + q_t}{p_t + 1}$$

$$+ \left[\frac{\dot{p}_t}{p_t} - \frac{\dot{q}_t}{q_t}\right]_{da_t=0} + \eta \left[\left(\frac{p_{t+dt}}{p_t} - \frac{q_{t+dt}}{q_t}\right) \frac{p_t + q_t}{p_{t+dt}+q_{t+dt}}\right]_{da_t\neq 0} = 0.$$

In order to get some intuitions of dynamics, let us first examine the local dynamics by phase diagram, assuming that the arrival of productivity switch is rare. We can solve for (22) and (24) for i_t and p_t with respect to y_t and q_t as:

$$(26a)\qquad i_t = \pi \frac{y_t - r(1-\theta)q_t}{\pi + r(1-\theta q_t)} \equiv i(y_t,q_t),$$

$$(26b)\qquad p_t = \frac{(1-\theta q_t)(y_t-rq_t) - \pi\theta q_t}{\pi + r(1-\theta q_t)} \equiv p(y_t,q_t).$$

In (26a), investment rate is an increasing function of output-capital ratio y_t. It is also an increasing function of capital price q_t for a given y_t, if and only if θ is large enough $(\pi+\delta)\theta > \pi$, so that the flow-of-funds effect of q_t on investment is larger than the crowding-out effect of increasing consumption with q_t. In (26b), land value-capital ratio is an increasing function of y_t and an decreasing function of q_t. If we substitute (26) into (23) and (25), we have a dynamical system with respect to output-capital ratio and Tobin's q as:

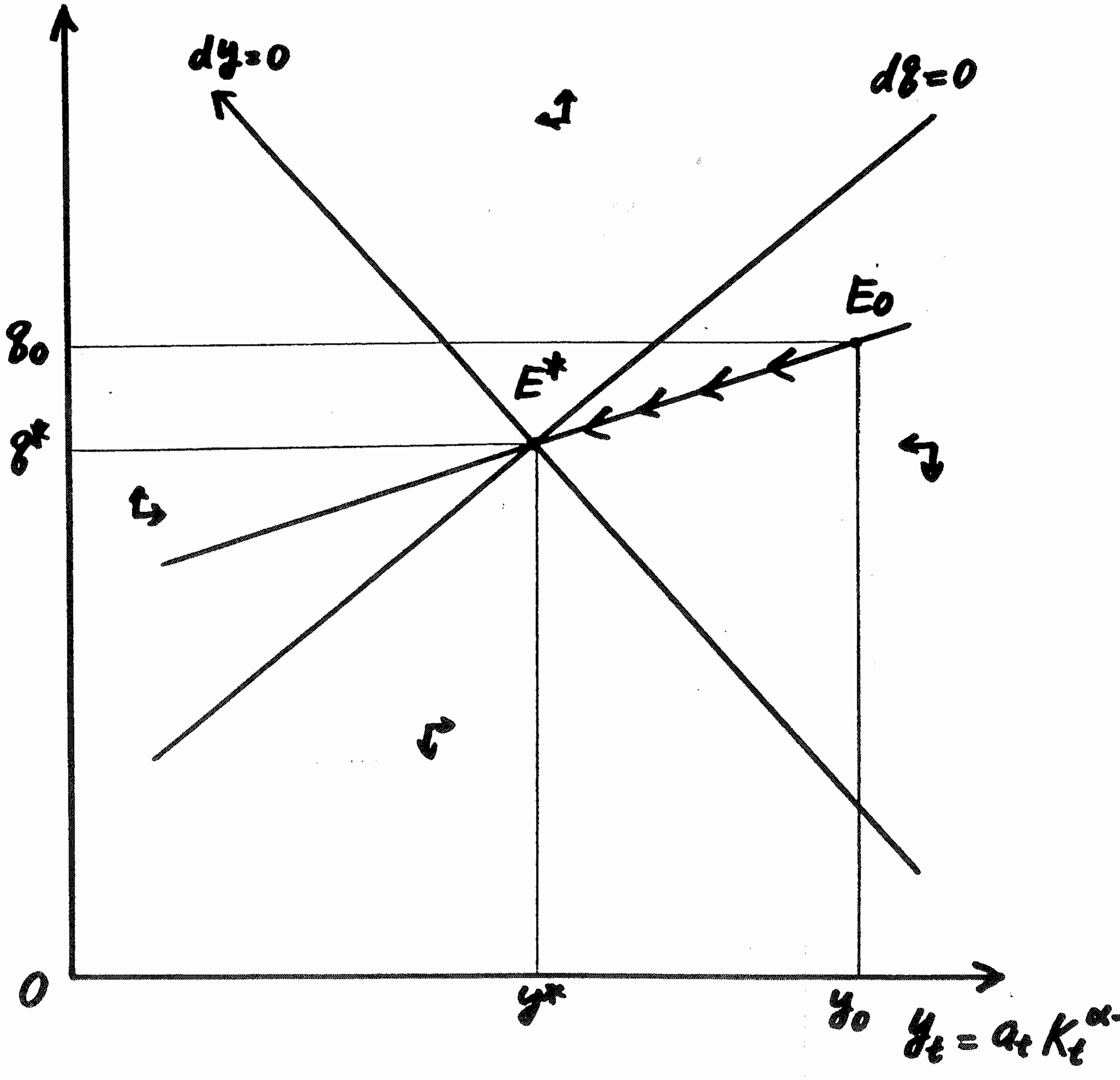
$$(27a)\qquad \frac{dy_t}{y_t} = \frac{da_t}{a_t} - (1-\alpha)\Big(i(y_t,q_t) - \delta \Big)\, dt \equiv H(y_t,q_t),$$

$$(27b)\quad (1-\varepsilon_{pq})\left[\frac{\dot{q}_t}{q_t}\right]_{da_t=0} = \left(\frac{1-\alpha}{p(y_t,q_t)} - \frac{\alpha}{q_t}\right)y_t + i(y_t,q_t) - (1-\alpha)\varepsilon_{py}[i(y_t,q_t)-\delta]$$

$$+ \pi \frac{q_t-1}{q_t}\frac{p(y_t,q_t)+q_t}{p(y_t,q_t)+1} + \eta\, [\varepsilon_{py}+(\varepsilon_{pq}-1)\varepsilon_{qa}]\left[\frac{da_t}{a_t}\right]_{da_t\neq 0} \equiv J(y_t,q_t),$$

where ε_{xz} is the elasticity of x with respect to z. (All the elasticities can be computed directly from (30), expect for $\varepsilon_{qa} \equiv \mu_q$ in (28c) later.) When the arrival rate of the productivity switch (η) is small, then we can analyze (27a) and (27b), using a phase diagram. We know $H_y < 0$ always, and $H_q < 0$ if and only if $(\pi+\delta)\theta > \pi$. When we assume that the elasticities are relatively constant, we also see $J_q > 0$, and that J_y is negative if effect of y on i is not too large because $(1-\alpha)/p < (\alpha/q)$. Then the typical phase diagram looks like in Figure 1.

Figure 1



There is a unique saddle point path converging to the steady state for a fixed aggregate productivity. Suppose that the economy is near the steady state for $a = a_{\ell}$, and that suddenly the aggregate productivity switches to high at $a = a_h$. Since the productivity switch is rare, this change is considered as largely unexpected. The output-capital ratio jumps from the steady state level y^* to a higher level y_0. Then capital price q_0 jumps up from q^* to q_0 on the saddle point path. Both output and investment increase. After the initial date, the capital will gradually accumulate over time with further increase in aggregate output and declining Tobin's q along the saddle-point path from E_0 to E^*, until the economy will converge to the neighborhood of the steady state.

An alternative way to analyze the dynamics is to examine a linearalized system in the neighborhood of the steady state. Using notation $\hat{x}_t$ as proportional deviation of variable x_t from the steady state value x, i.e., $\hat{x}_t \equiv (x_t - x)/x$, we can postulate the endogenous variables as functions of the state variables as:

(28a) $$\hat{i}_t = \lambda \hat{K}_t + \mu \hat{a}_t,$$

(28b) $$\hat{p}_t = \lambda_p \hat{K}_t + \mu_p \hat{a}_t,$$

(28c) $$\hat{q}_t = \lambda_q \hat{K}_t + \mu_q \hat{a}_t,$$

From (21) and (23), we know that:

(29) $$\hat{K}_t = \delta \hat{i}_t,$$

(30a) $$\left[\frac{\dot{p}_t}{p_t} - \frac{\dot{q}_t}{q_t} \right]_{da_t=0} = (\lambda_p - \lambda_q)\delta \hat{i}_t,$$

(30b) $$\left[\left(\frac{p_{t+dt}}{p_t} - \frac{q_{t+dt}}{q_t}\right) \frac{p_t + q_t}{p_{t+dt}+q_{t+dt}}\right]_{da_t\neq 0} = -2 (\mu_p - \mu_q) \hat{a}_t.$$

Thus linearizing (22),(24) and (25) around the steady state, we have:

(31a) $$-(1-\theta q)\delta\hat{i}_t + \pi p\hat{p}_t + \theta(\pi+\delta)q\hat{q}_t = 0,$$

(31b) $$\delta\hat{i}_t + rp\hat{p}_t + rq\hat{q}_t - y\hat{a}_t + (1-\alpha)y\hat{K}_t = 0,$$

(31c) $$\left[\frac{\alpha}{q}\hat{q}_t - \frac{1-\alpha}{p}\hat{p}_t\right]y + \left[\frac{1-\alpha}{p} - \frac{\alpha}{q}\right]y[\hat{a}_t-(1-\alpha)\hat{K}_t] + (1+\lambda_p-\lambda_q)\delta\hat{i}_t$$

$$+ \frac{\pi}{q}\left[\frac{p+q^2}{p+1}\hat{q}_t - p\left(\frac{q-1}{p+1}\right)^2\hat{p}_t\right] - 2\eta(\mu_p-\mu_q)\hat{a}_t = 0.$$

Substituting (28) into (31), these equations must hold for any $\hat{K}_t$ and $\hat{a}_t$. Thus we have 6 unknowns variables $(\lambda,\lambda_p,\lambda_q,\mu,\mu_p,\mu_q)$ which satisfy 6 equations, two terms of K_t and a_t in each of (31a),(31b) and (31c). With a bit of algebra, we can show that there is a unique negative λ which satisfies the saddle-point stability of the dynamical system. Also for a fairly mild regularity restrictions on parameters, we have

(32a) $$-1 < \lambda < 0, \quad -1 < \lambda_p < 0, \quad \lambda_q < 0, \quad \lambda_p < \lambda_q < \lambda_p + 1,$$

(32b) $$0 < \mu, \quad 0 < \mu_p, \quad 0 < \mu_q.$$

Equation (32a) says that aggregate investment (I_t) is an increasing function of capital stock, even if the investment rate (I_t/K_t) is a decreasing function of capital stock. Equation (32a) also implies that aggregate land value is an increasing function of capital, even if the ratio of land value to capital is a decreasing function of capital. It also says that Tobin's q is most likely a decreasing function of capital stock. Equation (32b) means that the investment, land value and capital value are all increasing functions of aggregate productivity. Together with (21) and (29), we can fully characterize the stochastic process of asset prices and aggregate quantities in the neighborhood of the steady state.

For applications of asset pricing, it is perhaps more natural to consider that the aggregate productivity follows a geometric Brownian motion, instead of two point Markov process:

$$(33) \qquad da_t = \sigma a_t dz_t,$$

where z_t is a Wiener process and $\sigma > 0$ is standard deviation of innovation of the aggregate productivity. Such an economy can be considered as the economy of [Merton(1975)] with illiquid capital stock. With this stochastic process of productivity, we no longer can presume that the investing agents are always credit constrained. Depending on the state of the economy, the investing agents may invest up to the maximum, may invest without credit constraint, or may not invest at all. In Appendix, we derive some very preliminary results about this economy.

III. Workers and Government

In this section we introduce workers and government. Suppose that output is now produced from homogeneous capital, land and labor, according to the aggregate production function:

$$(34) \qquad Y'_t = a'_t \; K_t^{\alpha'} \; L_t^{\beta'} \; N_t^{1-\alpha'-\beta'}, \qquad 0 < \alpha', \; \beta', \; 1-\alpha'-\beta'$$

where Y'_t is aggregate output, K_t is capital, L_t is land and N_t is labor. Suppose also that there is a continuum of a new group of people, called workers, with population size of unity. The workers supply labor, but they do not have productive investment opportunities. The utility of workers at date 0 is described as:

$$(35) \qquad \int_0^{\infty} u(c_t - \frac{\omega}{1+\gamma} N_t^{1+\gamma}) \; e^{-rt} \, dt, \qquad 0 < \gamma, \; \omega$$

where c_t is consumption, N_t is labor, and u() is instantaneous utility function which satisfies $u' > 0$, $u'' < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$. We also assume that the workers cannot borrow against their future wage income. We call the agents described in the Basic Model with occasional productive investment opportunities 'entrepreneurs.'

Beside workers and entrepreneurs, there is a large agent, called government. The government does not produce nor have productive investment opportunities. The government owns capital K_t^G and land L_t^G, taxes on workers T_t lump-sum, and purchases goods G_t. Denoting the government's total asset as B_t^G, the balance sheet condition and budget constraint of government are:

$$(36a) \qquad P_t \, L_t^G + q_t \, K_t^G = B_t^G,$$

$$(36b) \qquad dB_t^G = (R_t^L dt + dP_t)L_t^G + (R_t^K dt + dq_t)K_t^G + (T_t - G_t)dt.$$

If the government is not subject to the short-sales constraint because of its superior commitment technology, then the government can take a negative position on liquid asset, L_t^G by issuing government security whose return is identical to land. Thus the government can act as a banker.[11] We take the behavior of government as exogenous.

There is a competitive labor market in which the producers hire labor. Then, the real wage rate w_t will be equal to both marginal product of labor and marginal disutility of labor:

$$(37) \quad (1-\alpha'-\beta')a_t' K_t^{\alpha'} L_t^{\beta'} N_t^{1-\alpha'-\beta'} = w_t = \omega N_t^{\gamma}$$

Now we define the gross profit of producers Y_t as aggregate output minus aggregate wage for the workers. Then from (37), we now have:

$$(38) \quad Y_t \equiv Y_t' - w_t N_t = a_t K_t^{\alpha} L_t^{\beta}, \quad \text{where}$$

$$\alpha \equiv \alpha' \frac{1+\gamma}{\alpha'+\beta'+\gamma}, \quad \beta \equiv \beta' \frac{1+\gamma}{\alpha'+\beta'+\gamma}, \text{ and } a_t \equiv (\alpha'+\beta')\left(\frac{1-\alpha'-\beta'}{\omega}\right)^{\frac{1-\alpha'-\beta'}{\alpha'+\beta'+\gamma}} a_t'^{\frac{1+\gamma}{\alpha'+\beta'+\gamma}}.$$

Because there is no income effect of labor supply of workers, we can write down the gross profit function as a function of capital and land. Because the marginal disutility of labor is an increasing function of labor ($\gamma > 0$), the gross profit function is a decreasing returns to scale in capital and land, i.e., $\alpha + \beta < 1$.[12]

Concerning the entrepreneur's environment, the economy is exactly the same as Basic Model of the previous section. The only difference from the Basic Model is that, in perfectly competitive rental market for capital and

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[11] In [Kiyotaki-Moore(2000)], we consider the role of bankers who have resource-consuming commitment technology.

[12] If tax on workers is proportional to wage income at rate τ_t instead of lump-sum, then ω should be replaced by $\omega/(1-\tau_t)$ in (38), which reduces the aggregate gross profit productivity a_t. In text, we avoid the distortionary tax in order to concentrate on the analysis of liquidity.

land, the rental income per unit are equal to the marginal product of capital and land, taking into account the depreciation as:

$$(39a) \qquad R_t^K = \alpha' \frac{Y_t'}{K_t} - \delta q_t = \frac{\alpha}{\alpha+\beta} a_t K_t^{\alpha-1} - \delta q_t,$$

$$(39b) \qquad R_t^L = \beta' \frac{Y_t'}{L_t} = \frac{\beta}{\alpha+\beta} a_t K_t^{\alpha}.$$

In the following, we restrict the analysis in the neighborhood of the steady state equilibrium, by assuming the aggregate productivity shock is small as in (21). We also conjecture that the steady state economy is monetary as in Proposition 1. We later derive the condition which guarantees the economy to be monetary, after describing the competitive monetary equilibrium. Then, the investing entrepreneurs face credit constraints. Also, the equilibrium rates of returns on land and capital are both lower than the time preference rate. Then the workers will not save at all, and consume always the entire disposable wage income, $w_t N_t - T_t$, as long as the arrival of productivity switch is infrequent and the size of productivity shock is small so that the incentive for consumption smoothing is not too large. Here, the workers do not save, not because they are myopic or irrational, but because they do not expect future investment opportunity and because the rates of returns on land and capital are lower than the time preference rate in the monetary economy. In fact, the worker's time preference rate is the same with the entrepreneurs.

It is again convenient to describe the equilibrium in the intensive form. Define $k_t = (K_t - K_t^G)/K_t$: fraction of capital stock owned by private agents, $\ell_t = 1 - L_t^G$: fraction of land owned by private agents, $g_t = (G_t - T_t)/K_t$: the government fiscal primary deficit relative to aggregate capital stock, and $b_t^G = B_t^G/K_t$: the ratio of government asset to aggregate capital stock. Then the competitive equilibrium is described by:

$$(40) \qquad (1 - \theta q_t)\, i_t = \pi(p_t \ell_t + \theta q_t k_t),$$

(41) $$y_t = i_t + g_t + r(p_t\ell_t + q_tk_t),$$

(42) $$\left[\frac{\beta}{p_t} - \frac{\alpha}{q_t}\right]\frac{y_t}{\alpha+\beta} + i_t + \pi\,\frac{q_t-1}{q_t}\,\frac{p_t\ell_t + q_tk_t}{p_t\ell_t + k_t}$$

$$+ \left[\frac{\dot{p}_t}{p_t} - \frac{\dot{q}_t}{q_t}\right]_{da_t=0} + \eta\left[\left(\frac{p_{t+dt}}{p_t} - \frac{q_{t+dt}}{q_t}\right)\frac{p_t\ell_t + q_tk_t}{p_{t+dt}\ell_t + q_{t+dt}k_t}\right]_{da_t\neq0} = 0,$$

(43) $$p_t(1 - \ell_t) + q_t(1 - k_t) = b_t^G,$$

(44) $$db_t^G = \left[\left(\frac{\alpha}{\alpha+\beta}\,y_t - q_ti_t\right)(1-k_t) + \frac{\beta}{\alpha+\beta}\,y_t(1-\ell_t) - g_t\right]dt$$

$$+ (1-k_t)dq_t + (1-\ell_t)dp_t,$$

(45) $$\frac{dy_t}{y_t} = -(1-\alpha)(i_t-\delta)dt + \frac{da_t}{a_t}.$$

Equation (40) is flow-of-funds constraint of the investing entrepreneurs. Equation (41) describes the goods market clearing condition. Aggregate output is equal to the sum of investment, government purchase, and consumption of workers and entrepreneurs. Because the consumption of workers is equal to the disposable wage income, the goods market clearing is equivalent to the condition that the gross profit (aggregate output minus pre-tax wage income) is equal to investment, government primary deficit, and entrepreneur's consumption. Equation (42) is the optimal portfolio condition of the entrepreneurs. Since the workers do not hold assets, the entrepreneurs' assets are equal to the entire private assets. Equation (43) is the government's balance sheet condition, and (44) is the budget constraint of the government in intensive form. Equation (45) describes the evolution of the ratio of aggregate output to capital stock, due to capital accumulation and productivity change. Now the aggregate state of the economy is summarized by aggregate output-capital ratio, aggregate productivity, government total asset, fiscal policy, and portfolio policy (which satisfies the balance sheet condition): $s_t = (y_t, a_t, b_t^G, g_t, k_t, \ell_t)$. Competitive equilibrium is described by price functions $p_t = p(s_t)$, $q_t = q(s_t)$, and investment rate $i_t = i(s_t)$, and the evolutions of a_t, y_t and b_t^G, which satisfies (21) and (40-45).

Obviously, if the government reduces the lump-sum tax on workers by selling liquid land now, and if it will increase the tax in future in order to buy back land, then the real allocation will change, because the workers' consumption is equal to their disposable income. Thus, Ricardian Equivalence Theorem between lump-sum tax finance and debt fiance no longer holds here, because workers are credit constrained. A little less straightforward is the effect of governmenmt's portfolio policy, i.e., the monetary policy.

Suppose that the government does an open market operation to purchase capital by selling land (or the government liquid security whose return is identical to land.) Before analyzing the dynamic effect, let us analyze the long-run effect in the steady state. The steady state equilibrium is described by the corresponding equations with (40,41,42,44) with $i = \delta$ as:

$$(1 - \theta q)\delta = \pi(p\ell + \theta qk), \tag{46}$$

$$y = \delta + g + r(p\ell + qk), \tag{47}$$

$$\left(\frac{\beta}{p} - \frac{\alpha}{q}\right)\frac{y}{\alpha+\beta} + \delta + \pi\,\frac{q-1}{q}\,\frac{p\ell + qk}{p\ell + k} = 0, \tag{48}$$

$$g = \frac{\beta}{\alpha+\beta}\, y\,(1-\ell) + \left(\frac{\alpha}{\alpha+\beta}\, y - \delta q\right)(1-k) \tag{49}$$

The relationship between the primary fiscal deficit rate g, the government land holding $1-\ell$ and capital holding $1-k$ must satisfy the budget constraint of the government (49) in order to keep the total asset of the government constant at the steady state.

First, we can derive the condition under which the economy is monetary:

(Assumption 2) $\quad \theta < \theta^{**} \equiv 1 - \frac{1}{\delta+\pi k}\,\frac{\pi}{r}\left(\frac{\beta}{\alpha}(r+\delta) + r - g\right).$

Assumption 2 is a generalization of Assumption 1 for the economy with workers and government. In particular, the threshold faction of saleable capital at the time of investment is an increasing function of k, or a decreasing function of liquidity to private agents ℓ (due to the budget constraint of the government (49)). Thus, roughly speaking, the economy is more likely to be monetary, if a fraction of liquid land held by the private agents is small. In the following, we assume Assumption 2 so that the economy is monetary in the neighborhood of the steady state.

We now compare the two steady states for alternative government portfolio policies with the same primary fiscal deficit rate and the same aggregate productivity.[13] Suppose that a 'new' steady state corresponds to a government portfolio of more capital holding and less land holding than 'old' steady state. In other words, in 'new' steady steady state, the supply of assets to the private agents is more liquid in the sense that the private agents holds a larger fraction (ℓ) of liquid land and a smaller fraction (k) of illiquid capital. From equations (46-49), we can show that:

Proposition 2: Suppose that in the 'new' steady state, government holds more capital and less land than 'old' steady state for the same primary fiscal deficit ratio. Suppose also that both old and new steady state satisfy Assumption 2. Then, in the new steady state:

(i) Tobin's q is smaller;
(ii) liquidity premium is lower;

than the old steady state, if the size of government is not very large. Moreover, if the economy is not too far from the first best allocation, then;

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[13] The government total asset is unstable in (44), because the real rates of return on land and capital are positive in the neighborhood of steady state as in [Sargent and Wallace(1981)]. Thus, even if the productivity is constant, once-for-all an open market operation starting from a steady state economy will not converge to another steady state in the long run without adjustment of fiscal policy. The government needs to adjust g during the intermediate period in order to achieve the steady state with different portfolio with the same g in the long-run.

(iii) capital stock, employment and output are larger.

(Proof) in Appendix.

Intuitively, Proposition 2 implies that if the government holds more illiquid portfolio so that the private agent can hold more liquid portfolio, then the liquidity shortage of the economy is less severe in the steady state. Tobin's q is smaller, liquidity premium is smaller, and aggregate capital, employment and output are most likely to be larger. An interesting aspect is the valuation of capital, q, which is lower even if the private agents hold a smaller fraction of capital stock. This is because that the gap between Tobin's q and unity arises because of the credit constraint, and a larger supply of liquidity mitigates the shortage of liquidity and reduces Tobin's q towards one. A Modigliani-Miller Theorem for open market operation in [Wallace(1983)] breaks down here, because the government portfolio choice directly affects the ratio of the liquid asset to the illiquid asset of the private agents, and it is the private liquid assets holding which lubricates transferring resources from the savers to the investors. One the other hand, if Assumption 2 does not hold so that the economy achieves the first best allocation, then the rates of returns on liquid asset and illiquid capital would be equal, and the government open market operation would have no effect on the resource allocation.

Perhaps, one aspect that our model is at odd is the property that the workers do not save at all. If, instead of Poisson arrival of a productivity switch with infinitesimal size of shock, we have larger productivity shocks, or continual productivity shocks as in the case of geometric Brownian motion, then the workers will save in order to smooth consumption, despite that the expected rate of returns on assets are smaller than the time preference rate as in buffer stock model. (See [Carroll(1992)] and [Deaton(1992)].) However, the buffer stock model by itself does not explain well why the workers do not save in capital. Alternatively, we can extend the model to allow workers to have investment opportunities of small size.

Suppose that each worker suffer a 'health' shock according to a Poisson process with arrival rate ρ. With the shock, the worker has to spend ν unit of goods instantaneously in order to maintain his human capital (capacity to supply labor). After arrival of one shock, there is a time interval T, during which the worker is immune against shock and $wT >> \nu$ in the steady state. After this interval, the arrival rate of the shock goes back to ρ. The shock is assumed to be not contractible. We restrict the attention to the case in which ρ and ν are sufficiently small so that the economy continues to be monetary by a slightly different condition than Assumption 2.

Then, we can show that, for a small enough θ, (i) in normal time, each worker holds exactly ν units of liquid asset; (ii) with the arrival of a shock, the worker sells the entire liquid asset in order to meet the shock; (iii) the worker saves with the liquid asset at constant rate during the immune period in order to accumulate ν units at the end of the immune period; and (iv) the worker does not hold the illiquid capital. Intuitively, the worker does not save more than ν units of the liquid asset, because both the rate of returns on the illiquid and the liquid assets are lower than the time preference rate under the monetary economy. (See Proposition 1(ii).) The worker does not save with the illiquid capital even if the expected rate of return on capital dominates that of liquid land, because the worker needs more illiquid capital in order to meet the shock, which turns out to be 'more expensive,' as:

$$(50) \qquad \frac{1}{\theta}\left\{ r + \delta - E_t\left[\frac{R_t^K dt + dq_t}{q_t dt}\right] \right\} > r - E_t\left[\frac{R_t^L dt + dP_t}{P_t dt}\right].$$

Equation (50) implies that the opportunity cost of holding $1/\theta$ units of capital is larger than the opportunity cost of holding 1 unit of land, (remember that the agent can sell only θ fraction of capital in order to finance investment which includes overcoming the 'health' shock.) In the Appendix, we prove (50) holds in the neighborhood of steady state, if the economy is monetary and θ is small enough.

In the monetary economy, the expected returns on liquid asset (land) and illiquid capital both are lower than the time preference rate. Despite these low returns on assets, people save because people expect to face credit constraint when they need to finance the investment expenditure in future.

The entrepreneurs save substantial amounts of both liquid and illiquid asset, because their investment opportunity is large (or constant returns to scale here.) The workers save only small amount of liquid asset, because their investment opportunity is small (or fixed size here.) If the workers were expecting large investment opportunity in future, say, buying house or sending children to college, then these workers would save in substantial amount like entrepreneurs.[14] If people with large productive investment opportunities were not credit constrained, then their saving would not depend upon the expectation of future investment opportunities. Therefore, another unique feature of the monetary economy is that the agent's saving behavior depends upon his expectations of productive investment opportunities.

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[14] In national income account, when households build and renovate houses, it counts as investment because such households engage in entrepreneurial activity. We can regard these households with large investment opportunities as entrepreneurs instead of workers.

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Appendix 1: Discrete-Time Formulation of the Agent's Behavior

Here, we describe the individual agent's behavior using a discrete-time formulation, assuming that Tobin's q exceeds unity. The agent's utility is given by:

$$\text{(A1)} \qquad V_0 = E_0 \left(\sum_{t=0}^{\infty} \beta^t \ln c_t \right),$$

where c_t is consumption of period t and $0 < \beta < 1$ is the discount factor (which is not the same as the coefficient of production function in Section II.) Suppose that it takes one period for the invested capital to become productive:

$$\text{(A2)} \qquad K_t = \lambda K_{t-1} + I_t,$$

where λ is one minus one-period depreciation rate and K_t is capital stock at the end of period t which becomes available for production in the next period, thus $Y_t = a_t K_{t-1}^{\alpha}$. Let P_t and q_t be prices of land and capital after earning the rental income at period t, (so-called ex-dividend prices.)

When the agent holds ℓ_{t-1} units of land and k_{t-1} units of capital from the last period, the flow of funds constraint for the agent who has productive investment opportunity is:

$$\text{(A3)} \qquad c_t + P_t\ell_t + (1 - \theta q_t)i_t = (R_t^L + P_t)\ell_{t-1} + (R_t^K + \theta\lambda q_t)k_{t-1}.$$

When Tobin's q exceeds unity, the agent invests to the maximum and sells entire land:

$$\text{(A4)} \qquad k_t = (1-\theta)(i_t + \lambda k_{t-1}), \quad \text{and} \quad \ell_t = 0.$$

We can combine these as:

$$\text{(A5)} \qquad c_t + \frac{1-\theta q_t}{1-\theta} k_t = (R_t^L + P_t)\ell_{t-1} + (R_t^K + \lambda)k_{t-1} \equiv b_t,$$

where b_t is the value of total asset of the investing agent. The value of $(1-\theta q_t)/(1-\theta)$ is the cost of acquiring one unit of capital for the investing agent, because the investing agent who sells θ fraction of invested capital needs only downpayment $(1-\theta q_t)$ in order to acquire $1-\theta$ units of capital. The investing agent also values the remaining capital from the previous period in replacement cost λk_{t-1} instead of market value $\lambda q_t k_{t-1}$ due to the limited liquidity of capital.

For the agent who does not have investment opportunity in period t, the flow-of-funds constraint is:

$$\text{(A6)} \quad c_t + P_t\ell_t + q_tk_t = (R^L_t + P_t)\ell_{t-1} + (R^K_t + \lambda q_t)k_{t-1} \equiv b_t.$$

The value of capital from the previous period is $\lambda q_t k_{t-1}$ to him, because he is not credit constrained. Let $V_i(b_t, s_t)$ be the value of an agent with an investment opportunity when he has b_t units of total asset and the state of the economy is s_t. Let $V_u(b_t, s_t)$ be the value of an agent without an investment opportunity in period t. Let Π be probability that each agent has an investment opportunity in each period. Then, Bellman equations are:

$$\text{(A7)} \quad V_i(b_t,s_t) = \underset{k_t}{\text{Max}} \left\{ \ln\left(b_t - \frac{1-\theta q_t}{1-\theta}k_t\right) \right.$$
$$\left. + \beta\, E_t\left[\Pi V_i((R^K_{t+1}+\lambda)k_t, s_{t+1}) + (1-\Pi)V_u((R^K_{t+1}+\lambda q_{t+1})k_t, s_{t+1})\right] \right\},$$

$$\text{(A8)} \quad V_u(b_t,s_t) = \underset{\ell_t, k_t}{\text{Max}} \left\{ \ln(b_t - P_t\ell_t - q_tk_t) + \beta E_t\left[\Pi V_i((R^L_{t+1}+P_{t+1})\ell_t + (R^K_{t+1}+\lambda)k_t, s_{t+1})\right.\right.$$
$$\left.\left. + (1-\Pi)V_u((R^L_{t+1}+P_{t+1})\ell_t + (R^K_{t+1}+\lambda q_{t+1})k_t, s_{t+1})\right] \right\}.$$

We guess that $c_t = (1-x)b_t$. Then the first order conditions for consumption and portfolio choice are:

$$\text{(A9)} \quad \frac{1}{(1-x)b_t} = \frac{1-\theta}{1-\theta q_t}\beta E_t\left(\frac{\Pi(R^K_{t+1} + \lambda)}{(1-x)(R^K_{t+1}+\lambda)\frac{1-\theta}{1-\theta q_t}xb_t} + \frac{(1-\Pi)(R^K_{t+1} + \lambda q_{t+1})}{(1-x)(R^K_{t+1}+\lambda q_{t+1})\frac{1-\theta}{1-\theta q_t}xb_t} \right)$$

$$\text{(A10)} \quad \frac{1}{(1-x)b_t}$$

$$= \frac{1}{P_t}\,\frac{1}{xb_t(1-x)}\,\beta E\left(\frac{\Pi(R^L_{t+1} + P_{t+1})}{[(R^t_{t+1}+P^L_{t+1})\ell_t+(R^K_{t+1}+\lambda)k_t]} + \frac{(1-\Pi)(R^L_{t+1} + P_{t+1})}{[(R_{t+1}+P^L_{t+1})\ell_t+(R_{t+1}+\lambda q^K_{t+1})k_t]}\right)$$

$$= \frac{1}{q_t}\,\frac{1}{xb_t(1-x)}\,\beta E_t\left(\frac{\Pi(R^K_{t+1} + \lambda)}{[(R^L_{t+1}+P_{t+1})\ell_t+(R^K_{t+1}+\lambda)k_t]} + \frac{(1-\Pi)(R^K_{t+1} + \lambda q_{t+1})}{[(R^L_{t+1}+P_{t+1})\ell_t+(R^K_{t+1}+\lambda q_{t+1})k_t]}\right)$$

From these and (A6), we see that $x = \beta$, or $c_t = (1-\beta)b_t$. Also a bit of algebra from (A10), we get:

$$\text{(A11)}\ 0 \;=\; E_t\left(\left(\frac{R^L_{t+1}+P_{t+1}}{P_{t+1}} - \frac{R^K_{t+1}+\lambda q_{t+1}}{q_t}\right)\frac{P_t\ell_t + q_tk_t}{(R^L_{t+1}+P_{t+1})\ell_t+(R^K_{t+1}+\lambda q_{t+1})k_t}\right.$$

$$\left. + \;\Pi\,\frac{\lambda\,\dfrac{q_{t+1}-1}{q_t}\,\dfrac{R^L_{t+1}+P_{t+1}}{P_t}\,(P_t\ell_t+q_tk_t)^2}{[(R^L_{t+1}+P_{t+1})\ell_t+(R^K_{t+1}+\lambda q_{t+1})k_t]\,[(R^L_{t+1}+P_{t+1})\ell_t+(R^K_{t+1}+\lambda)k_t]}\right).$$

The first line is the gap in the expected rates of returns on land and capital in terms of utility, without taking into account the return with investment opportunity. The second line is the expected advantage of liquid land over illiquid capital in financing downpayment for the investment in terms of utility. This advantage is proportional to the probability of meeting productive investment opportunity times the expected gap between Tobin's q and unity in the next period.

Let us redefine the length of period Δ instead of 1, consider R^L_t and R^K_t as rental income of land and capital per unit of time. Dividing (A11) by Δ, we get:

$$\text{(A11)}\quad 0 = E_t\left(\left(\frac{R^L_{t+\Delta}\Delta+P_{t+\Delta}-P_t}{P_{t+\Delta}\ \Delta} - \frac{R^K_{t+\Delta}\Delta+\lambda q_{t+\Delta}-q_t}{q_t\ \Delta}\right)\frac{P_t\ell_t + q_tk_t}{(R^L_{t+\Delta}\Delta+P_{t+\Delta})\ell_t+(R^K_{t+\Delta}\Delta+\lambda q_{t+\Delta})k_t}\right.$$

$$\left. + \frac{\Pi}{\Delta}\ \frac{\lambda\ \dfrac{q_{t+\Delta}}{q_t} - 1\ \dfrac{R^L_{t+\Delta}\Delta+P_{t+\Delta}}{P_t}\ (P_t\ell_t+q_tk_t)^2}{[(R^L_{t+\Delta}\Delta+P_{t+\Delta})\ell_t+(R^K_{t+\Delta}\Delta+\lambda q_{t+\Delta})k_t][(R^L_{t+\Delta}\Delta+P_{t+\Delta})\ell_t+(R^K_{t+\Delta}\Delta+\lambda)k_t]}\right).$$

Let $\beta = e^{-r\Delta}$, $\lambda = e^{-\delta\Delta}$, $\Pi = 1 - e^{-\pi\Delta}$. When we take a limit of both sides as Δ converges to 0, we get equation (13) in the text. Similarly for consumption, we have $c_t\Delta = (1-\beta)b_t$. Thus, we get $c_t = rb_t$, as $\Delta \to 0$.

Appendix 2: Proof of Propositions

Proof of Proposition 1: Using the flow-of-funds constraint of investing agents (20a), the goods market clearing condition (20b) and asset market clearing condition (20c) become:

$$\text{(A12)} \quad \phi(q,K;\theta) \equiv \delta(\pi+r) + [\pi - \theta(\pi+\delta)]rq - \pi a K^{\alpha-1} = 0,$$

$$\text{(A13)} \quad \psi(q,K;\theta) \equiv -aK^{\alpha-1}\left[\frac{\alpha}{q} - \frac{\pi(1-\alpha)}{\delta-(\pi+\delta)\theta q}\right] + \delta + \frac{\pi}{\pi+\delta}\frac{q-1}{q}\frac{\delta+[\pi-\theta(\pi+\delta)]q}{1-\theta q} = 0.$$

Combining these, the steady-state value of capital price q solves

$$\text{(A14)} \quad -\left(\delta(\pi+r)+[\pi-\theta(\pi+\delta)]rq\right)\left[\frac{\alpha}{\pi q} - \frac{1-\alpha}{\delta-(\pi+\delta)\theta q}\right] + \delta + \frac{\pi}{\pi+\delta}\frac{q-1}{q}\frac{\delta+[\pi-\theta(\pi+\delta)]q}{1-\theta q}$$

$$\equiv F(q;\theta) = 0.$$

Proof of Proposition (i): From (A14), we see F is an increasing function of q. Then there is a unique q which is larger than 1 if and only if $F(1;\theta) < 0$. Because F is an increasing function of θ, (noting the value of [ ] in the first term is negative,) $F(1;\theta) < 0$, if and only if Assumption 1 holds.

Proof of Proposition 1(ii): From the optimal portfolio condition (20c), the returns on land is dominated by capital:

$$\text{(A15)} \quad \frac{R^L}{P} \equiv \frac{1-\alpha}{P} aK^{\alpha} < \frac{\alpha}{q} aK^{\alpha-1} - \delta \equiv \frac{R^K}{q} - \delta,$$

if and only if q exceeds unity, or Assumption 1 holds. Also the rate of returns on capital is lower than the time preference rate, $(R^K/q) - \delta < r$, if and only if:

$$\text{(A16)} \quad q > \frac{\alpha\delta(\pi+r)}{\pi\delta + r[(1-\alpha)\pi+\alpha\theta(\pi+\delta)]} \equiv \underline{q},$$

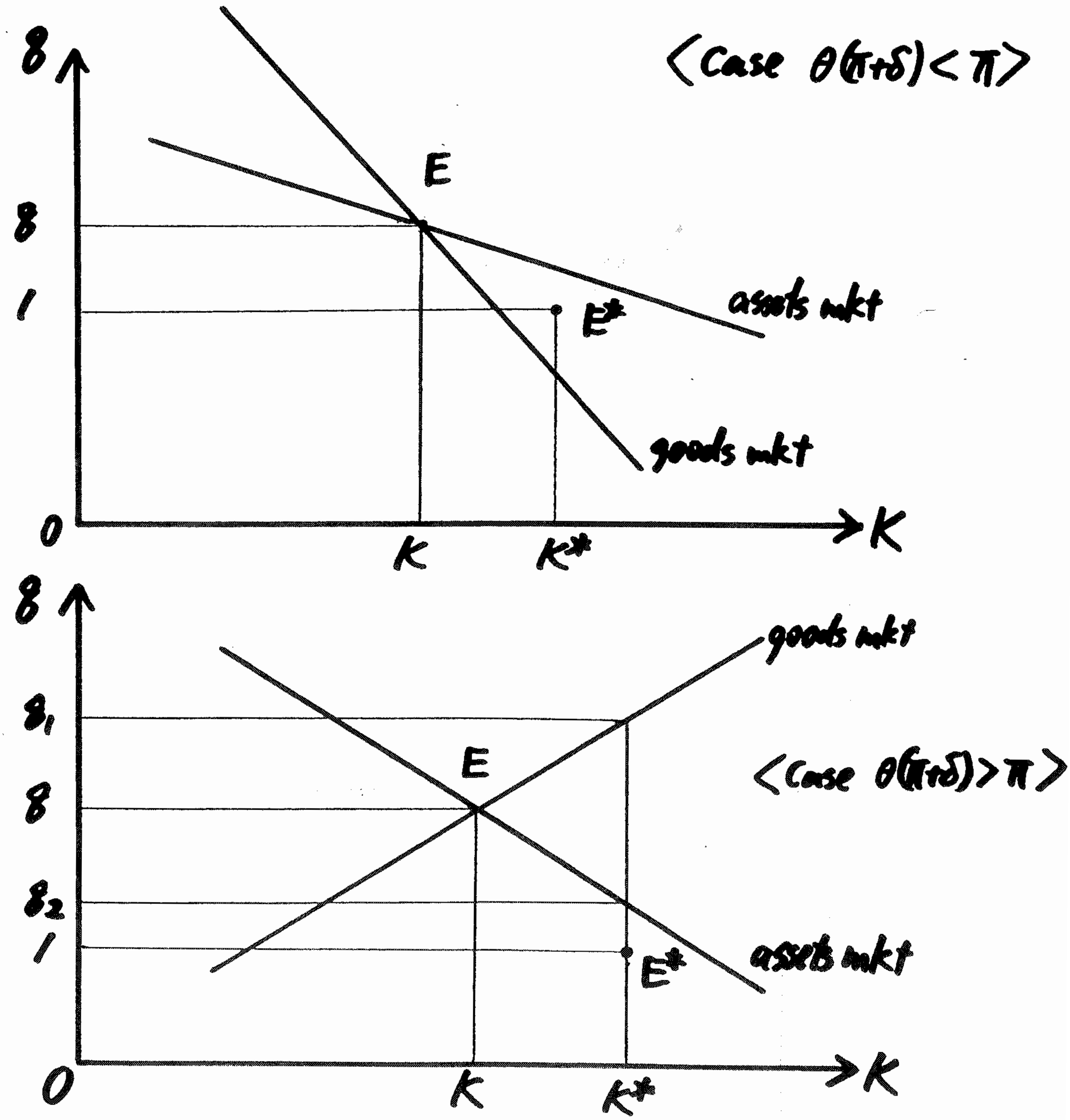
This is equivalent to $F(\underline{q};\theta) < 0$ in equation (A14), which holds if Assumption 1 is satisfied.

Proof of Proposition 1(iii): Substituting expression of the first best capital stock K^* in Proposition 1, we have:

(A17a) $\phi(1, K^*) > 0,$

(A17b) $\psi(1, K^*) < 0.$

Because we already know $q > 1$ in the monetary equilibrium, we see the relationship between the monetary equilibrium and the first best allocation as in Figure 2.



Then there is under-capital accumulation in monetary economy, $K < K^*$, for the case in which $\pi > \theta(\pi+\delta)$. For the case of $\pi < \theta(\pi+\delta)$, $K < K^*$, if and only if $q_1 > q_2$, where q_1 and q_2 solves $\phi(q_1,K^*) = 0$ and $\psi(q_2,K^*) = 0$. A a bit of algebra shows that $\psi(q_1,K^*;\theta) > \psi(q_1,K^*;\theta^*) = 0$, (which implies $q_1 > q_2$), if Assumption 1 holds and $\pi < \theta(\pi+\delta)$. *(Q.E.D).*

<u>Proof of Proposition 2:</u> Combining equations (20) in the text, we can get the steady-state market clearing condition for goods market and asset market as:

$$\text{(A18a)} \qquad \Phi(q,y;\theta,k) \equiv (\pi+r)\delta + rq[(1-\theta)\pi k - \theta\delta] + \pi g - \pi y = 0,$$

$$\text{(A18b)} \qquad \Psi(q,y;\theta,k) \equiv \pi \frac{(1 - \frac{\alpha}{\alpha+\beta}k)y-g-(1-k)\delta q}{\delta - \theta(\delta+\pi k)q} - \frac{\alpha}{\alpha+\beta}\frac{y}{q} + \delta$$
$$+ \frac{\pi}{\delta+\pi k}\frac{q-1}{q}\frac{\delta+q[(1-\theta)\pi k-\theta\delta]}{1 - \theta q} = 0.$$

Combining these, we also get:

$$\text{(A19)} \qquad \tilde{F}(q;\theta,k) = \frac{\pi g+(\pi+r)+rq[(1-\theta)\pi k-\theta\delta]}{\alpha + \beta}\left(\frac{\alpha+\beta-\alpha k}{\delta-\theta q(\delta+\pi k)} - \frac{\alpha}{\pi q}\right) + \delta$$
$$- \pi\frac{g+(1-k)\delta q}{\delta-\theta q(\delta+\pi k)} + \frac{\pi}{\delta+\pi k}\frac{q-1}{q}\frac{\delta+q[(1-\theta)\pi k-\theta\delta]}{1-\theta q} = 0.$$

From (A19), we can see $\tilde{F}_q > 0$, $\tilde{F}_\theta > 0$, and $\tilde{F}_k < 0$, if g is close to 0 (balanced fiscal budget) and k is not too far from 1 (government does not own too much capital.)

Proof of Proposition 2(i): From (A19), we have:

$$\text{(A20)} \qquad \frac{\partial q}{\partial k} = -\frac{\tilde{F}_k}{\tilde{F}_q} > 0,$$

Thus, a reduction of k with the open market operation will reduce Tobin's q.

Proof of Proposition 2(ii): From the asset market equilibrium (A18a), the liquidity premium is equal to:

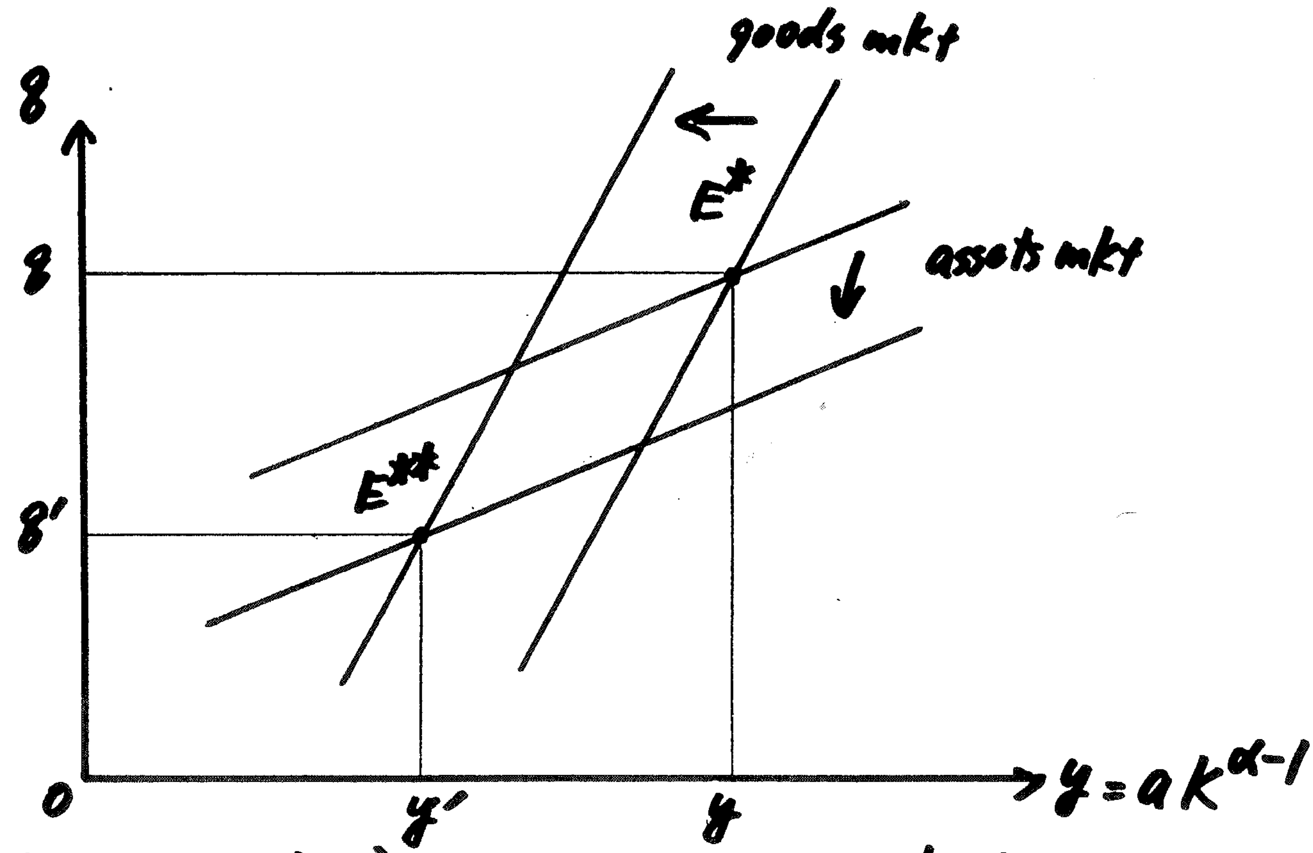
$$\text{(A21)} \qquad \frac{R^K}{q} - \delta - \frac{R^L}{P} = \frac{\pi}{\delta+\pi k}\frac{q-1}{q}\frac{\delta+q[(1-\theta)\pi k-\theta\delta]}{1 - \theta q} \equiv \varphi(k,q).$$

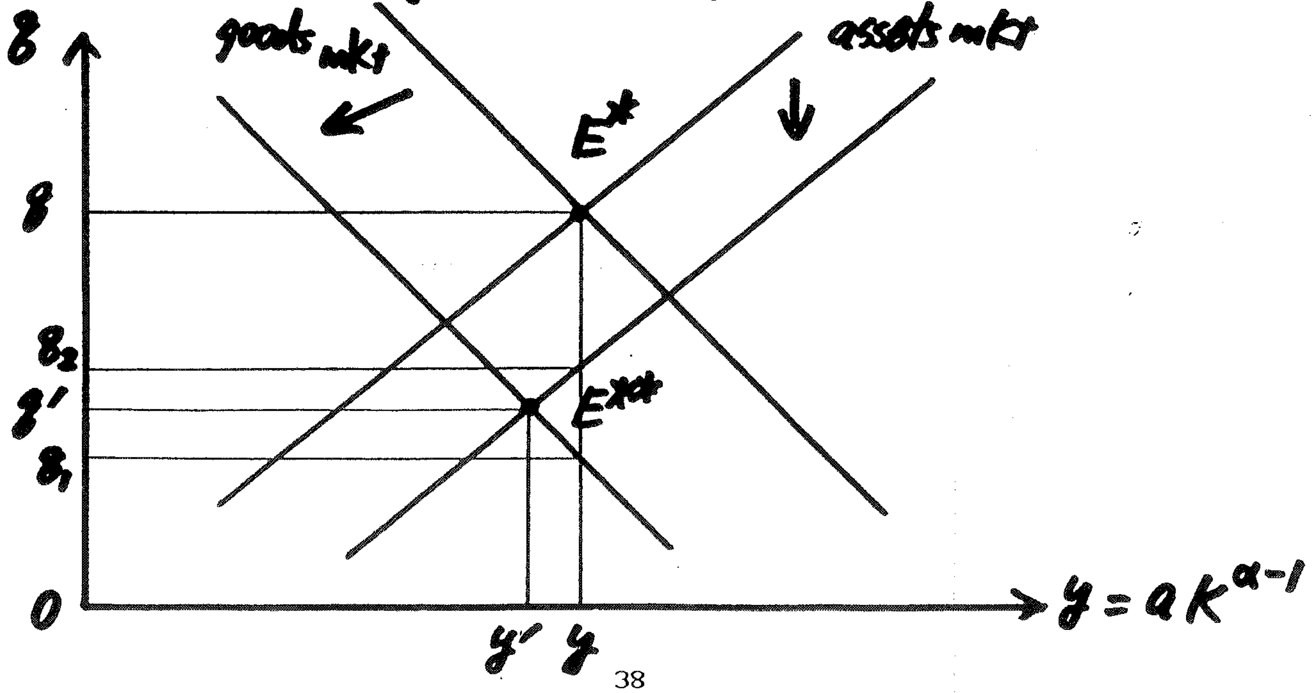
Then we have

(A22) $$\frac{d\varphi}{dk} = \varphi_k + \varphi_q \frac{dq}{dk} > 0,$$

because φ_k, $\varphi_q > 0$ and $\frac{dq}{dk} > 0$ from (A20). Thus the open market operation will reduce the liquidity premium.

Proof of Proposition 2(iii): From (A18), market clearing conditions of goods market and asset market are described in Figure 3.





From (A18), we have $\Phi_k > 0$. Also $\Psi_k < 0$, if the government is small (g is close to 0 and k is close to 1) and θ is not too far from θ^{**}. Also from (A18), we know $\Phi_q < 0$ if and only if $\theta(\pi k+\delta) > \pi k$. We also have $\Phi_y < 0$, $\Psi_q > 0$, and $\Psi_y < 0$ if the government is small. Thus the open market operation shifts the equilibrium schedules as in Figure 3. For the case of $\theta(\pi k+\delta) < \pi k$, we know y will decreases or capital stock will increase in the long run with the open market operation. For the case of $\theta(\pi k+\delta) > \pi k$, we can show that $q_1 < q_2$ with a bit of algebra, and thus that K will increase in the long-rum with the open market operation. *Q.E.D.*

<u>Proof of Inequality (50):</u> The inequality (50) holds if and only if:

$$r - \frac{\beta}{\alpha+\beta}\frac{y}{p} \quad < \quad \frac{1}{\theta}\left(r + \delta - \frac{\alpha}{\alpha+\beta}\frac{y}{q} \right),$$

from (39). When the government is negligible, g=0 and k=1, then this is equivalent to

$$\left(\alpha\delta - \theta q[\alpha(\pi+\delta)+\beta\pi]\right)\left((\pi+r)\delta + [\pi-\theta(\pi+\delta)]rq\right) \quad < \quad \pi(\alpha+\beta)q[\delta+(1-\theta)r][\delta-\theta(\pi+\delta)q],$$

which we can verify from (A14) with a bit of algebra, if θ is small enough. *Q.E.D.*

Appendix 3: Basic Model with Brownian Productivity Shock

In this appendix, we layout the basic model with aggregate productivity shock which follows the geometric Brownian process as in (33). We ignore workers and government here. We first guess that the aggregate output-capital ratio $y_t = a_t K_t^{\alpha-1}$ and the aggregate productivity a_t summarize the aggregate state of nature. We also guess that land price - capital ratio, capital price, and investment rate as functions of y_t only:

$$\text{(A23)} \qquad p_t = p(y_t), \quad q_t = q(y_t), \quad \text{and } i_t = i(y_t).$$

Then we derive the equilibrium conditions which these functions must satisfy, and then verify that our guess is correct.

Using Ito's lemma with (15) and (16), we have:

$$\text{(A24)} \qquad \frac{dy_t}{y_t} = (\alpha-1)[i(y_t) - \delta]\, dt + \sigma\, dz_t$$

Let us write down the rates of returns on land and capital as:

$$\text{(A25a)} \qquad \frac{R^L dt + dP}{P} = \frac{(1-\alpha)ydt + dp(y)}{p(y)} + [i(y)-\delta]dt \equiv R^P(y)\, dt + \omega^P(y)\, dz,$$

$$\text{(A25b)} \qquad \frac{R^K dt + dq}{q} = \frac{\alpha ydt + dq(y)}{q(y)} - \delta dt \equiv R^q(y)\, dt + \omega^q(y)\, dz.$$

Let b_t be the total asset as in (5) and let m_t be the share of liquid asset in portfolio, $m_t \equiv P_t \ell_t / b_t$. Then the budget constraint of the agent is

$$\text{(A26)} \quad db_t = \left([m_t R_t^P + (1-m_t)R_t^q]b_t - c_t \right) dt + [m_t \omega_t^P + (1-m_t)\omega_t^q]b_t\, dz_t + \text{Max}\left(\frac{1-\theta}{1-\theta q_t}(1-m_t+m_t q_t)b_t - b_t,\ 0 \right) dM_t,$$

Thus Bellman equation of the agent is:

(A27) $$r\,V(b_t,y_t) = \max_{c_t,m_t} \Big\{ \ln c_t + V_b(b_t,y_t)\Big([m_tR^P_t + (1-m_t)R^q_t]b_t - c_t\Big)$$

$$+ \frac{1}{2} V_{bb}(b_t,y_t)[m_t\omega^P_t + (1-m_t)\omega^q_t]^2\, b_t^{\,2}$$

$$+ \pi \max\left[V\left(\frac{1-\theta}{1-\theta q_t}(1-m_t+m_tq_t)b_t, y_t\right) - V(b_t,y_t),\ 0 \right] \Big\}$$

The optimal investment rule is the same as Tobin's q theory of investment as in text. From the first.order conditions, we get optimal consumption rule and optimal portfolio rule as:

(A28) $$c_t = r\,b_t,$$

(A29) $$R^P_t - R^q_t - (\omega^P_t-\omega^q_t)[m_t\omega^p_t + (1-m_t)\omega^q_t] + \pi\,\frac{\mathrm{Max}(q_t-1,0)}{1-m_t+m_tq_t} = 0.$$

These are very similar to consumption rule (12) and portfolio rule (13) in the text, except for the effect of risk due to different stochastic process of productivity shock. From (A29), we can see the optimal share of liquid asset is:

(A30) $$m_t = m(R^p_t-R^q_t,\ q_t,\ \pi,\ \omega^p_t,\ \omega^q_t), \text{ where}$$

$$m_{(R^p-R^q)} > 0,\quad m_q \geq 0,\quad m_\pi \geq 0,\quad m_{\omega^P} < 0 \text{ and } m_{\omega^q} > 0.$$

The difference from the standard portfolio theory is that the liquid asset ratio is a weakly increasing function of Tobin's q and the arrival rate of productive investment opportunity.

Now we can combine the individual's behavior with market clearing conditions and define the equilibrium as price functions p(y),q(y) and investment rate i(y) which satisfy:

(A31) $$[1 - \theta q(y)]\; i(y) \leq \pi\,[p(y) + \theta q(y)], \text{ where}$$

$$\begin{cases} = \text{ holds, if } q(y) > 1, \\ \leq \text{ holds, if } q(y) = 1, \\ i(y) = 0, \text{ if } q(y) < 1. \end{cases}$$

$$\text{(A32)} \qquad y = i(y) + r\,[p(y) + q(y)],$$

$$\text{(A33)} \qquad [R^P(y) - R^q(y)] - [\omega^P(y) - \omega^q(y)]\frac{p(y)\omega^P(y)+q(y)\omega^q(y)}{p(y)+q(y)}$$

$$+ \pi\,\frac{\text{Max }[q(y)-1,0]}{q(y)}\,\frac{p(y)+q(y)}{p(y)+1} = 0,$$

where the returns characteristics $R^P(y)$, $R^q(y)$, $\omega^P(y)$ and $\omega^q(y)$ are defined in (A25). The stochastic process of a_t and y_t follow (33) and (A24). Equation (A31) describes the behavior of the aggregate investment. Equation (A32) describes the goods market equilibrium, and (A33) describes the asset market equilibrium. In (A33), the first term is the difference of the expected rates of returns on the liquid asset and illiquid capital, the second is the effect of risk aversion, and the last is the expected advantage of the liquid asset over illiquid capital for financing the productive investment. This last term distinguishes our model from a standard capital asset pricing model. All the above equilibrium conditions are functions of the output-capital ration y_t only, and the aggregate productivity and output-capital ratio are the sufficient statistics of the aggregate state of the economy. Thus our initial conjecture was correct.

This system is not much more complicated than the real business cycles model, or [Merton(1975)]. Thus, in principle, we can simulate the above system to examine the dynamics. Alternatively, if p(y) and q(y) were three-times differentialble, then we would know from Ito's Lemma that:

$$R^p(y) = \frac{(1-\alpha)y}{p(y)} + \left(1 + \frac{yp'(y)}{p(y)}(\alpha-1)\right)[i(y)-\delta] + \frac{1}{2}\sigma^2\frac{y^2p''(y)}{p(y)},$$

$$R^q(y) = \frac{\alpha y}{q(y)} - \delta + \frac{yq'(y)}{q(y)}(\alpha-1)[i(y)-\delta] + \frac{1}{2}\sigma^2\frac{y^2q''(y)}{q(y)},$$