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Liquidity, Business Cycles and Monetary Policy

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Liquidity, Business Cycles, and Monetary Policy*

Nobuhiro Kiyotaki and John Moore†

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Abstract

The paper presents a model of a monetary economy where there are differences in liquidity across assets. Money circulates because it is more liquid than other assets, not because it has any special function. There is a spectrum of returns on assets, reflecting their differences in liquidity. The model is used, first, to investigate how aggregate activity and asset prices fluctuate with shocks to productivity and liquidity; second, to examine what role government policy might have through open market operations that change the mix of assets held by the private sector. With its emphasis on liquidity rather than sticky prices, the model harks back to an earlier interpretation of Keynes (1936), following Tobin (1969).

JEL Classification: E10, E44, E50

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1 Introduction

This paper presents a model of a monetary economy where there are differences in liquidity across assets. Our aim is to study how aggregate activity and asset prices fluctuate with recurrent shocks to productivity and liquidity. In doing so, we examine what role government policy might have through open market operations that change the mix of assets held by the private sector.

Part of our purpose is to construct a workhorse model of money and liquidity that does not stray too far from the other workhorse of modern macroeconomics, the real business cycle model. We thus maintain the assumption of competitive markets. In a standard competitive framework, money has no role unless endowed with a special function, for example that the purchase of goods requires cash in advance. In our model, the reason why money can improve resource allocation is not because money has a special function but because, crucially, we assume that other assets are partially illiquid, less liquid than money. Ours might be thought of as a liquidity-in-advance framework.

Iliquidity has to do with some impediment to the resale of assets. With this in mind, we construct a model in which the resale of assets is a central feature of the economy. We consider a group of entrepreneurs, who each uses his or her own capital stock and skill to produce output from labor (which is supplied by workers). Capital depreciates and is restocked through investment, but the investment technology, for producing new capital from output, is not commonly available: in each period only some of the entrepreneurs are able to invest, and the arrival of investment opportunities is randomly distributed across entrepreneurs through time. Hence in each period there is a need to channel output from those entrepreneurs who don't have an investment opportunity (that period's savers) to those who do (that period's investors).

To acquire output for the production of new capital, an investing entrepreneur issues equity claims to the capital's future returns. However, we assume that because the investing entrepreneur's skill will be needed to produce these future returns and he cannot precommit to work, at the time of investment he can credibly pledge only a fraction – say θ – of the future returns from the new capital. Unless θ is high enough, he faces a borrowing constraint: he must finance part of the cost of investment from his available resources. The lower is θ, the tighter is the borrowing constraint: the larger is the downpayment per unit of investment that he must make out of his own funds.

He will typically have on his balance sheet two kinds of asset that can be resold to raise funds. He may have money. And he may have equity previously issued by other entrepreneurs. Both of these will have been acquired by him at some point in the past, when he himself was a saver.

Crucially, we suppose that equity is less liquid than money. We parameterize the degree to which equity is illiquid by making a stylized assumption: in each period only a proportion – say ϕ – of an agent's equity holding can be resold. Think of peeling an onion slowly, layer by layer, a fraction ϕ per period. Although the entrepreneur with an investment opportunity this period can readily divest ϕ of his equity holding, to divest any more he will have to wait until next period, by which time the opportunity may have disappeared. The lower is ϕ, the tighter is the resaleability constraint. Unlike his equity holding, the entrepreneur's money holding is perfectly liquid: it can all be used to buy goods straightaway.

In practice, of course, there are wide differences in resaleability across different kinds of equity: compare the stock of publicly-traded companies with shares in privately-held businesses. Indeed there are many financial assets that are hardly any less liquid than money, e.g., government bonds. Thus in our stylized model, "money" should be interpreted very broadly to include all financial assets that are essentially as liquid as money. Under the heading of "equity" come all financial assets that are less than perfectly liquid. By assumption, all these non-monetary assets are subject to the common resaleability constraint parameterized by ϕ.

To understand how fiat money can lubricate this economy, notice that the task of channelling goods from those entrepreneurs who don't have an investment opportunity into the hands of those who do is thwarted by the fact that investors are unable to offer savers adequate compensation: the borrowing constraint (θ) means that new capital investment cannot be entirely self-financed by issuing new equity, and the resaleability constraint (ϕ) means that sufficient of the old equity cannot change hands quickly. Fiat money can help alleviated this problem. Our analysis shows that if θ and ϕ aren't high enough – if (and only if) a particular combination of θ and ϕ lies below a certain threshold – then the circulation of fiat money, passing each period from investors to savers in exchange for goods, serves to boost aggregate activity. Whenever fiat money plays this essential role we say that the economy is a monetary economy. Whether or not agents use fiat money

– whether or not the economy is monetary – is determined endogenously.

We show that in a monetary economy, the expected rate of return on money is very low, less than the expected rate of return on equity. (The steady-state of an economy where the stock of fiat money is fixed would necessarily have a zero net return on money.) Nevertheless, a saver chooses to hold some money in his portfolio, because in the event that he has an opportunity to invest in the future he will be liquidity constrained, and money is more liquid than equity. The gap between the return on money and the return on equity is a liquidity premium.

We also show that both the returns on equity and money are lower than the rate of time preference. This is because borrowing constraints starve the economy of means of saving – too little equity can be credibly pledged – which raises asset prices and lowers yields. As a consequence, agents who never have investment opportunities, such as the workers, choose to hold neither equity nor money. Assuming workers cannot borrow against their future labor income, they simply consume their wage, period by period. This may help explain why certain households neither save nor participate in asset markets. It isn't that they don't have access to those markets, or that they are particularly impatient, but rather that the return on assets isn't enough to attract them. (The model can be extended to show that if workers face their own liquidity shocks then they may save, but only use money to do so.)

In our θ-ϕ framework, θ and ϕ are exogenous parameters. Although the borrowing constraint (θ) and the resaleability constraint (ϕ) might both be thought of as varieties of liquidity constraint,[1] in this paper we will be especially concerned with the effects of shocks to ϕ, which we identify as liquidity shocks. We are motivated here by the fact that in the recent financial turmoil many assets – such as asset-backed securities and auction-rate bonds – that used to be highly liquid became much less resaleable.[2] Even though we will focus on shocks to ϕ, it is important to recognize that θ is an essential

[1]Brunnermeier and Pedersen (2009) use "funding liquidity" to refer to the borrowing constraint and "market liquidity" to refer to the resaleability constraint.

[2]In our first presentations of this research (see, for example, Kiyotaki and Moore, 2001), although we separately identified the borrowing and resaleability constraints, for analytical convenience we set $\phi = \theta$. However, it helps to keep ϕ distinct from θ, as we do in the current paper, because we are thus able to pin down the effects of shocks to ϕ and identify a monetary policy that can be used in response.

We made use of the θ-ϕ framework in other papers, though sometimes with different notation: Kiyotaki and Moore (2002, 2003, 2005a and 2005b).

component of the model. Were θ to be sufficiently close to 1, then new capital investment could be self-financed by issuing new equity and there would be no need for old equity to circulate (reminiscent of the idea that in the Arrow-Debreu framework markets need open only once, at an initial date); liquidity shocks, shocks to ϕ, would have no effect.

The mechanism by which liquidity shocks affect our monetary economy is absent from most real business cycle models. In our model, there is a critical feedback from asset prices to aggregate activity. Consider a persistent liquidity shock: suppose ϕ falls and is anticipated to recover only slowly. The impact of this fall in resaleability is to shrink the funds available to investors to use as downpayment. Further, anticipating lower future resaleability, the price of equity falls relative to the value of money – think of this as a "flight to liquidity" – which tends to raise the size of the required downpayment per unit of investment. All in all, via this feedback mechanism, investment falls as ϕ falls. Asset prices and aggregate activity are vulnerable to liquidity shocks, unlike in a standard general equilibrium asset pricing model.

Our basic model, presented in Sections 2 and 3, has a fixed stock of fiat money. Government is introduced in the full model of Section 4, which examines monetary policy. This model bears some resemblance to a Keynesian IS-LM model, except that in our model prices and wages are fully flexible, and agents are maximizing.

How might government, through interventions by the central bank, ameliorate the effects of liquidity shocks? Specifically, how might policy change behavior in the private economy?

The central bank can costlessly change the supply of fiat money. However, in our framework with flexible prices, a once-and-for-all change in the money supply by means of a lump-sum transfer to the entrepreneurs – a helicopter drop – wouldn't have any effect on aggregate real variables.

The central bank can also buy and hold private equity – albeit that we impose the same upper bound (ϕ) on the rate at which it can divest. Unlike a helicopter drop, an open-market operation to purchase equity by issuing fiat money will affect aggregate activity. We show that the policy works by shifting up the ratio of the values of money to equity held by the private sector; cf. Metzler (1951). Investing entrepreneurs are in a position to invest more when their portfolios are more liquid. In effect, the government improves liquidity in the private economy by taking relatively illiquid assets onto its own books. This unconventional form of monetary policy has been employed by central banks around the world in recent years to ease the financial crisis, and appears to have met with some success. Interventions by the central bank have real effects in our economy because they operate across a liquidity margin - the difference in liquidity between money and equity; cf. Tobin (1969).

It is revealing to contrast this countercyclical policy, used to offset liquidity shocks, with the procyclical policy our model prescribes for dealing with productivity shocks. We find that a central bank should issue money to buy equity in high productivity states because it is precisely in those states that bottlenecks in the financial market between savers and investors – the binding borrowing and resaleability constraints – matter most.

Before we come to this policy analysis, it helps to start with the basic model without government. We will relate our paper to the literature and make some final remarks in Section 5. Proofs are contained in the Appendix.

2 The Basic Model without Government

Consider an infinite-horizon, discrete-time economy with four objects traded: a nondurable output, labor, equity and fiat money. Fiat money is intrinsically useless, and is in fixed supply M in the basic model of this and the next section.

There are two populations of agents, entrepreneurs and workers, each with unit measure. Let us start with the entrepreneurs, who are the central actors in the drama. At date t, a typical entrepreneur has expected discounted utility

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \tag{1}$$

of consumption path $\{c_t, c_{t+1}, c_{t+2}, ...\}$, where $u(c) = \log c$ and $0 < \beta < 1$. He has no labor endowment. All entrepreneurs have access to a constant-returns-to-scale technology for producing output from capital and labor. An entrepreneur holding k_t capital at the start of period t can employ ℓ_t labor to produce

$$y_t = A_t (k_t)^{\gamma} (\ell_t)^{1-\gamma} \tag{2}$$

output, where $0 < \gamma < 1$. Production is completed within the period t, during which time capital depreciates to λk_t, $0 < \lambda < 1$. We assume that the productivity parameter, $A_t > 0$ which is common to all entrepreneurs, follows a stationary stochastic process. Given that each entrepreneur can employ labor at a competitive real wage rate, w_t, gross profit is proportional to the capital stock:

$$y_t - w_t \ell_t = r_t k_t, \tag{3}$$

where, as we will see, gross profit per unit of capital, r_t, depends upon productivity, aggregate capital stock and labor supply.

The entrepreneur may also have an opportunity to produce new capital. Specifically, at each date t, with probability π he has access to a constant-returns technology that produces i_t units of capital from i_t units of output. The arrival of such an investment opportunity is independently distributed across entrepreneurs and through time, and is independent of aggregate shocks. Again, investment is completed within the period t – although newly-produced capital does not become available as an input to the production of output until the following period t+1:

$$k_{t+1} = \lambda k_t + i_t. \tag{4}$$

We assume there is no insurance market against having an investment opportunity.[3] We also make a regularity assumption that the subjective discount factor is larger than the fraction of capital left after production (one minus the depreciation rate):

$$Asssumption\ 1 : \beta > \lambda.$$

This mild restriction is not essential, but will make the distribution of capital and asset holdings across of individual entrepreneurs well-behaved.

In order to finance the cost of investment, the entrepreneur who has an investment opportunity can issue equity claims to the future returns from newly produced capital. Normalize one unit of equity at date t to be a claim to the future returns from the one unit of investment of date t: it pays r_{t+1} output at date t+1, λr_{t+2} at date t+2, $\lambda^2 r_{t+3}$ at date t+3, and so on.

We make two critical assumptions. First, the entrepreneur who produces new capital cannot fully precommit to work with it, even though his specific

[3] This assumption can be justified in a variety of ways. For example, it may not be possible to verify that someone has an investment opportunity; or verification may take so long that the opportunity has gone by the time the claim is paid out. A long-term insurance contract based on self-reporting does not work here because the people are able to trade assets covertly. Each of these justifications warrants formal modelling. But we are reasonably confident that even if partial insurance were possible our broad conclusions would still hold. So rather than clutter up the model, we simply assume that no insurance scheme is feasible.

skills will be needed for it to produce output. To capture this lack of commitment power in a simple way, we assume that an investing entrepreneur can credibly pledge at most a fraction $\theta < 1$ of the future returns.[4] Loosely put, we are assuming that only a fraction θ of the new capital can be mortgaged.

We take θ to be an exogenous parameter: the fraction of new capital returns that can be issued as equity at the time of investment. The smaller is θ, the tighter is the borrowing constraint that an investing entrepreneur faces. To meet the cost of investment, he has to use any money that he may hold, and raise further funds by – as far as possible – reselling any holding of other entrepreneurs' equity that he may have accumulated through past purchases.

The second critical assumption is that entrepreneurs cannot dispose of their equity holdings as quickly as money. Again to capture this idea in a simple way, we assume that, before the investment opportunity disappears, the investing entrepreneur can resell only a fraction $\phi_t < 1$ of his holding of other entrepreneurs' equity. (He can use all his money.) This is tantamount to assuming a peculiar transaction cost per period: zero for the first fraction ϕ_t of equity sold, and then infinite.

Like θ, we take ϕ_t to be an exogenous parameter: the fraction of equity holdings that can be resold in each period. The smaller is ϕ_t, the less liquid is equity; the tighter is the resaleability constraint.

We suppose that the aggregate productivity A_t and the liquidity of equity ϕ_t jointly follow a stationary Markov process in the neighborhood of the constant unconditional mean (A, ϕ). A shock to A_t is a productivity shock, and a shock to ϕ_t is a liquidity shock. (We do not shock θ, which is why it does not have a subscript.)

In general, an entrepreneur has three kinds of asset in his portfolio: money; his holding of other entrepreneurs' equity; and the uncommitted fraction, $1 - \theta$, of the returns from his own capital, which might loosely be termed "unmortgaged capital stock" - own capital stock minus own equity issued.

[4] Cf. Hart and Moore (1994), where the borrowing constraint is shown to be a consequence of the fact that the human capital of the agent who is raising funds – here, the investing entrepreneur – is inalienable.

Balance sheet	
money holding	own equity issued
holding of other entrepreneurs' equity	
own capital stock	net worth

It turns out to be in general hard to analyze aggregate fluctuations of the economy with these three assets, because there is a complex dynamic interaction between the distribution of asset holdings across the entrepreneurs and their choices of consumption, investment and portfolio. Thus, we make a simplifying assumption: in every period, we suppose that an entrepreneur can issue new equity against a fraction ϕ_t of any uncommitted returns from his old capital – in loose terms, he can mortgage a fraction ϕ_t of any as-yet-unmortaged capital stock. Think of mortgaging old capital stock – or reselling equity – as akin to peeling an onion slowly, layer by layer, a fraction ϕ_t in each period t.

The upshot of this assumption is that an entrepreneur's holding of others' equity and his unmortgaged capital stock are perfect substitutes as means of saving for him: both pay the same return stream per unit (r_{t+1} at date t+1, λr_{t+2} at date t+2, $\lambda^2 r_{t+3}$ at date t+3, ...); and up to a fraction ϕ_t of both can be resold/mortgaged per period. In effect, by making the simplifying assumption we have reduced down to two the number of assets that we need keep track of: besides money, the holdings of other entrepreneurs' equity ("outside equity") and the unmortgaged capital stock ("inside equity") can be lumped together simply as "equity".

Let n_t be the equity and m_t the money held by an individual entrepreneur at the start of period t. He faces two "liquidity constraints":

$$n_{t+1} \geq (1-\theta)i_t + (1-\phi_t)\lambda n_t, \text{ and} \tag{5}$$

$$m_{t+1} \geq 0. \tag{6}$$

During the period, the entrepreneur who invests i_t can issue at most θi_t equity against the new capital. And he can dispose of at most a fraction ϕ_t of his equity holding, after depreciation. Inequality (5) brings these constraints together: his equity holding at the start of period t+1 must be at least $1-\theta$ times investment plus $1-\phi_t$ times depreciated equity. Inequality (6) says that his money holding cannot be negative.

Let q_t be the price of equity in terms of output, the numeraire. q_t is also equal to Tobin's q: the ratio of the market value of capital to the replacement cost. Let p_t be the price of money. (Warning! p_t is customarily defined as the inverse: the price of general output in terms of money. But, a priori, money may not have value, so better not to make it the numeraire.) The entrepreneur's flow of funds constraint at date t is then given by

$$c_t + i_t + q_t(n_{t+1} - i_t - \lambda n_t) + p_t(m_{t+1} - m_t) = r_t n_t. \tag{7}$$

The left-hand side (LHS) is his expenditure on consumption, investment and net purchases of equity and money. The right-hand side (RHS) is his dividend income, which is proportional to his holding of equity at the start of this period.

Turn now to the workers. At date t, a typical worker has expected discounted utility

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} U\left[c'_s - \frac{\omega}{1+\nu}\left(\ell'_s\right)^{1+\nu}\right], \tag{8}$$

of consumption path $\left\{c'_t, c'_{t+1}, c'_{t+2}, ..\right\}$ given his labor supply path $\{\ell'_t, \ell'_t, \ell'_t, ..\}$, where $\omega > 0, \nu > 0$ and $U[\cdot]$ is increasing and strictly concave. The flow-of-funds constraint of the worker is

$$c'_t + q_t(n'_{t+1} - \lambda n'_t) + p_t(m'_{t+1} - m'_t) = w_t \ell'_t + r_t n'_t. \tag{9}$$

The consumption expenditure and net purchase of equity and money in the LHS is financed by wage and dividend income. Workers do not have investment opportunities, and cannot borrow against their future labor income.

$$n'_{t+1} \geq 0, \text{ and } m'_{t+1} \geq 0. \tag{10}$$

An equilibrium process of prices $\{p_t, q_t, w_t\}$ is such that: entrepreneurs choose labor demand ℓ_t to maximize gross profit (3) subject to the production function (2) for a given start-of-period capital stock, and they choose consumption, investment, capital stock and start-of-next-period equity and money holdings $\{c_t, i_t, k_{t+1}, n_{t+1}, m_{t+1}\}$, to maximize (1) subject to (4) - (7); workers choose consumption, labor supply, equity and money holding $\left\{c'_t, \ell'_t, n'_{t+1}, m'_{t+1}\right\}$ to maximize (8) subject to (9) and (10); and the markets for general output, labor, equity and money all clear.

Before we characterize equilibrium, it helps to clear the decks a little by suppressing reference to the workers. Given that their population has unit measure, it follows from (8) and (9) that their aggregate labor supply equals $(w_t/\omega)^{1/\nu}$. Maximizing the gross profit of a typical entrepreneur with capital k_t, we find his labor demand, $k_t\left[(1-\gamma)A_t/w_t\right]^{1/\gamma}$ which is proportional to k_t. So if the aggregate stock of capital at the start of date t is K_t, labor-market clearing requires that

$$(w_t/\omega)^{1/\nu} = K_t\left[(1-\gamma)A_t/w_t\right]^{1/\gamma}.$$

Substituting back the equilibrium wage w_t into the LHS of (3), we find that the individual entrepreneur's maximized gross profit equals $r_t k_t$ where

$$r_t = a_t\left(K_t\right)^{\alpha-1}, \tag{11}$$

and the parameters a_t and α are derived from A_t, γ, ω and ν:

$$\begin{aligned} a_t &= \gamma\left(\frac{1-\gamma}{\omega}\right)^{\frac{1-\gamma}{\gamma+\nu}}\left(A_t\right)^{\frac{1+\nu}{\gamma+\nu}} \\ \alpha &= \frac{\gamma(1+\nu)}{\gamma+\nu}. \end{aligned} \tag{12}$$

Note from (12) that α lies between 0 and 1, so that r_t – which is parametric for the individual entrepreneur – declines with the aggregate stock of capital K_t, because the wage increases with K_t. But for the entrepreneurial sector as a whole, gross profit $r_t K_t$ increases with K_t. Also note from (12) that r_t is increasing in the productivity parameter A_t through a_t. Later we will show that in the neighborhood of the steady state monetary equilibrium, a worker will choose to hold neither equity nor money. That is, the worker simply consumes his labor income at each date:

$$c_t' = w_t \ell_t'. \tag{13}$$

We are now in a position to characterize the equilibrium behavior of the entrepreneurs. Consider an entrepreneur holding equity n_t and money m_t at the start of period t. First, suppose he has an investment opportunity: let this be denoted by a superscript i on his choice of consumption, and start-of-next-period equity and money holdings, $\left(c_t^i, n_{t+1}^i, m_{t+1}^i\right)$. He has two ways of acquiring equity n_{t+1}^i: either produce it at unit cost 1, or buy it in the market at price q_t. (See the LHS of the flow-of-funds constraint (7), where, recall, i_t corresponds to investment). If q_t is less than 1, the agent will not invest. If q_t equals 1, he will be indifferent. If q_t is greater than 1, he will invest by selling as much equity as he can subject to the constraint (5). The entrepreneur's production choice is similar to Tobin's q theory of investment.

As the aggregate productivity and liquidity of equity (A_t, ϕ_t) follow a stochastic process in the neighborhood of constant (A, ϕ), we have the following claim in the neighborhood of the steady state equilibrium (all the proofs are in the Appendix):

Claim 1 *Suppose that θ and ϕ satisfy*

$$Condition\ 1 : (1-\lambda)\theta + \pi\lambda\phi > (1-\lambda)(1-\pi).$$

Then in the neighborhood of the steady state:

(i) the allocation of resources is first best;

(ii) Tobin's q is equal to unity: $q_t = 1$;

(iii) money has no value: $p_t = 0$;

(iv) the gross dividend is roughly equal to the time preference rate plus the depreciation rate: $r_t \simeq \frac{1}{\beta} - \lambda$.

If the investing entrepreneurs can issue new equity relatively freely and/or existing equity is relatively liquid – Condition 1 is satisfied – then the equity market transfers enough resources from the savers to the investing entrepreneurs to achieve the first best allocation.[5] There is no advantage to having investment opportunity; Tobin's q is equal to 1 (the market value of capital is equal to the replacement cost) and both investing entrepreneurs and savers earn the same net rate of return on equity – approximately equal to the time preference rate. (Note that the usual risk premium is almost negligible in the first best with our logarithmic utility function). Because the economy achieves the first best allocation without money, money has no value.

In the following we want to restrict attention to an equilibrium in which q_t is greater than 1. We also want money to have value in equilibrium. Let

[5] In steady state, aggregate saving (which equals aggregate investment) is equal to the depreciation of capital. The RHS of Condition 1 is the ratio of the aggregate saving of the (fraction $1-\pi$) non-investing entrepreneurs to the aggregate capital stock in first best. The LHS is the ratio of the equity issued/resold by the investing entrepreneurs to the aggregate capital stock: $\theta(1-\lambda)$ corresponds to new equity issued and $\pi\lambda\phi$ corresponds to old equity resold by the (fraction π) investing entrepreneurs. Thus Condition 1 says that the equity issued/resold by the investing entrepreneurs is enough to shift the aggregate saving of the non-investing entrepreneurs.

us assume that θ and ϕ satisfy:

$$\textit{Assumption } 2: 0 < \Phi(\theta, \phi), \text{ where}$$

$$\begin{aligned}\Phi(\theta,\phi) \equiv\ & \pi\lambda\beta^2(1-\pi)(1-\phi)[(1-\lambda)(1-\pi)-(1-\lambda)\theta-\pi\lambda\phi] \\ & +[(\beta-\lambda)(1-\pi)-(1-\lambda)\theta-\pi\lambda\phi]\,[1-\lambda+\pi\lambda-(1-\lambda)\theta-\pi\lambda\phi] \\ & \cdot[\lambda(1-\beta)(1-\pi)+(1-\lambda)\theta+\lambda(\beta+\pi-\pi\beta)\phi].\end{aligned}$$

Observe all the terms in the RHS are positive, except for the terms $(1-\lambda)(1-\pi)-(1-\lambda)\theta-\pi\lambda\phi$ and $(\beta-\lambda)(1-\pi)-(1-\lambda)\theta-\pi\lambda\phi$. Thus a sufficient condition for Assumption 2 is

$$(1-\lambda)\theta+\pi\lambda\phi < (\beta-\lambda)(1-\pi),$$

and a necessary condition is

$$(1-\lambda)\theta+\pi\lambda\phi < (1-\lambda)(1-\pi).$$

Notice that if Condition 1 in Claim 1 were satisfied, then this necessary condition would not satisfied and there could be no equilibrium with valued fiat money. Under Assumption 2, however, the upper bound on θ and ϕ is tight enough to ensure that the following claim holds.

Claim 2 *Under Assumption 2, in the neighborhood of the steady state:*
(i) the price of money, p_t, is strictly positive;
(ii) the price of capital, q_t, is strictly greater than 1;
(iii) an entrepreneur with an investment opportunity faces the binding liquidity constraints and will not choose to hold money: $m^i_{t+1}=0$.

We will be in a position to prove the claim once we have laid out the equilibrium conditions – we use a method of guess-and-verify in the following. For completeness, it should be pointed out that for intermediate values of θ and ϕ which satisfy neither Assumption 2 nor Condition 1, we can show that money has no value even though the liquidity constraint (5) still binds. To streamline the paper, we have chosen not to give an exhaustive account of the equilibria throughout the parameter space.

There is a caveat to Claim 2(i). Fiat money can only be valuable to someone if other people find it valuable, hence there is always a non-monetary equilibrium in which the price of fiat money is zero. Thus when there is a monetary equilibrium in addition to the non-monetary equilibrium, we restrict attention to the monetary equilibrium: $p_t > 0$. Claim 2(iii) says that the entrepreneur prefers investment with the maximum leverage to holding money, even though the return is in the form of equity which at date t+1 is less liquid than money. (Incidentally, even though the investing entrepreneurs don't want to hold money for liquidity purposes, the non-investing entrepreneurs do – see below. This is why Claim 2(i) holds).

Thus, for an investing entrepreneur, the liquidity constraints (5) and (6) are both binding. His flow of funds constraint (7) can be rewritten

$$c_t^i + (1 - \theta q_t)\, i_t = (r_t + \lambda \phi_t q_t)\, n_t + p_t m_t. \tag{14}$$

In order to finance investment i_t, the entrepreneur issues equity $\theta\ i_t$ at price q_t. Thus the second term in the LHS is the investment cost that has to be financed internally: the downpayment for investment. The LHS equals the total liquidity needs of the investing entrepreneur. The RHS corresponds to the maximum liquidity supplied from dividends, sales of the resaleable fraction of equity after depreciation and the value of money. Solving this flow-of-funds constraint with respect to the equity of the next period, we obtain

$$\begin{aligned} c_t^i + q_t^R n_{t+1}^i &= r_t n_t + [\phi_t q_t + (1 - \phi_t) q_t^R] \lambda n_t + p_t m_t, \qquad (15) \\ \text{where } q_t^R &\equiv \frac{1 - \theta q_t}{1 - \theta} < 1, \text{ as } q_t > 1. \qquad (16) \end{aligned}$$

The value of q_t^R is the effective replacement cost of equity to the investing entrepreneur: because he needs a downpayment $1 - \theta q_t$ for every unit of investment of which he retains $1 - \theta$ inside equity, he needs $(1 - \theta q_t)/(1 - \theta)$ to acquire one unit of inside equity. The RHS of (15) is his net worth: gross dividend plus the value of his depreciated equity λn_t – of which the resaleable fraction ϕ_t is valued at market price and the non-resaleable fraction $1 - \phi_t$ is valued by the effective replacement cost – plus the value of money.

Given the discounted logarithmic preferences (1), the entrepreneur saves a fraction β of his net worth, and consumes a fraction $1-\ \beta$:[6]

$$c_t^i = (1 - \beta) \left\{ r_t n_t + [\phi_t q_t + (1 - \phi_t) q_t^R] \lambda n_t + p_t m_t \right\}. \tag{17}$$

[6] Compare (1) to a Cobb-Douglas utility function, where the expenditure share of present consumption out of total wealth is constant and equal to $1/\left(1 + \beta + \beta^2 + ...\right) = 1 - \beta$.

And so, from (14), we obtain an expression for his investment in period t:

$$i_t = \frac{(r_t + \lambda\phi_t q_t)\, n_t + p_t m_t - c_t^i}{1 - \theta q_t}. \tag{18}$$

Investment is equal to the ratio of liquidity available after consumption to the required downpayment per unit of investment.

Next, suppose the entrepreneur does not have an investment opportunity: denote this by a superscript s to stand for a saver. The flow-of-funds constraint (7) reduces to

$$c_t^s + q_t n_{t+1}^s + p_t m_{t+1}^s = r_t n_t + q_t \lambda n_t + p_t m_t. \tag{19}$$

For the moment, let us assume that constraints (5) and (6) do not bind. Then the RHS of (19) corresponds to the entrepreneur's net worth. It is the same as the RHS of (15), except that now his depreciated equity is valued at the market price, q_t . From this net worth he consumes a fraction $1-\ \beta$:

$$c_t^s = (1-\beta)(r_t n_t + q_t \lambda n_t + p_t m_t). \tag{20}$$

Note that consumption of an entrepreneur who does not have investment opportunity is larger than consumption of an investing entrepreneur if both hold the same equity and money at the start of period. The remainder is split across a savings portfolio of m_{t+1} and n_{t+1}.

To determine the optimal portfolio, consider the choice of sacrificing one unit of consumption c_t to purchase either $1/p_t$ units of money or $1/q_t$ units of equity, which are then used to augment consumption at date t+1. The first-order condition is

$$\begin{aligned} u'(c_t) &= E_t\left\{\frac{p_{t+1}}{p_t}\beta\left[(1-\pi)\, u'\left(c_{t+1}^s\right) + \pi u'(c_{t+1}^i)\right]\right\} \\ &= (1-\pi)\, E_t\left\{\frac{r_{t+1} + \lambda q_{t+1}}{q_t}\beta u'\left(c_{t+1}^s\right)\right\} \\ &\quad + \pi E_t\left\{\frac{r_{t+1} + \lambda\phi_{t+1} q_{t+1} + \lambda\left(1-\phi_{t+1}\right) q_{t+1}^R}{q_t}\beta u'\left(c_{t+1}^i\right)\right\}. \end{aligned} \tag{21}$$

The RHS of the first line of (21) is the expected gain from holding $1/p_t$ additional units of money at date t+1: money always yields p_{t+1} which, proportionately, will increase utility by $u'\left(c_{t+1}^s\right)$ when he does not have a date t+1 investment opportunity (probability $1-\pi$) and by $u'(c^i_{t+1})$ when he does (probability π). The second line is the expected gain from holding $1/q_t$ additional units of equity at date t+1. Per unit, this additional equity yields r_{t+1} dividend plus its depreciated value. With probability $1-\pi$ the entrepreneur does not have a date t+1 investment opportunity, the depreciated equity is valued at the market price, q_{t+1}, and these yields increase utility in proportion to $u'\left(c^s_{t+1}\right)$. With probability π the entrepreneur does have an investment opportunity at date t+1, in which case he will value depreciated equity by the market price q_{t+1} for the resaleable fraction and by the effective replacement cost q^R_{t+1} for the non-resaleable fraction, and these yields increase utility in proportion to $u'\left(c^i_{t+1}\right)$.

Notice that because the effective replacement cost is lower than the market price, the effective return on equity is lower just when the entrepreneur is more in need of funds, viz. when an investment opportunity arises and his marginal utility of consumption is higher ($c^i_{t+1} < c^s_{t+1}$). That is, over and above aggregate risk, equity carries an idiosyncratic risk: its effective return is negatively correlated with the idiosyncratic variations in marginal utility that stem from the stochastic investment opportunities. Money is free from such idiosyncratic risk.

We are now in a position to consider the aggregate economy. The great merit of the expressions for an investing entrepreneur's consumption and investment choices, c^i_t and i_t, and a non-investing entrepreneurs' consumption and savings choices, c^s_t, n_{t+1} and m_{t+1}, is that they are all linear in start-of-period equity and money holdings n_t and m_t.[7] Hence aggregation is easy: we do not need to keep track of the distributions. Notice that, because workers do not choose to save, the aggregate holdings of equity and money of the entrepreneurs are equal to aggregate capital stock K_t and money supply M. At the start of date t, a fraction π of K_t and M is held by entrepreneurs who have an investment opportunity. From (18), total investment, I_t, in new capital therefore satisfies

$$(1-\theta q_t)\,I_t = \pi\left\{\beta\left[(r_t+\lambda\phi_t q_t)K_t + p_t M\right] - (1-\beta)(1-\phi_t)\lambda q^R_t K_t\right\}. \quad (22)$$

Goods market clearing requires that total output (net of labor costs, which equal the consumption of workers), $r_t K_t$, equals investment plus the

[7] From (19) and (20), the value of savings, $q_t n^s_{t+1} + p_t m^s_{t+1}$, is linear in n_t and m_t, and (the reciprocal of) the portfolio equation (21) is homogeneous in $\left(n^n_{t+1}, m^n_{t+1}\right)$, noting that $u'(c) = 1/c$ given the logarithmic utility function.

consumption of entrepreneurs. Using (17) and (20), we therefore have

$$\begin{aligned} r_t K_t &= a_t K_t^{\alpha} = I_t + (1-\beta) \cdot \\ & \left\{ [r_t + (1-\pi+\pi\phi_t)\lambda q_t + \pi\,(1-\phi_t)\,\lambda q_t^R] K_t + p_t M \right\}. \end{aligned} \tag{23}$$

It remains to find the aggregate counterpart to the portfolio equation (21). During period t, the investing entrepreneurs sell a fraction θ of their investment I_t, together with a fraction ϕ_t of their depreciated equity holdings $\pi\lambda K_t$, to the non-investing entrepreneurs. So the stock of equity held by the group of non-investing entrepreneurs at the end of the period is given by $\theta I_t + \phi_t \pi\lambda K_t + (1-\pi)\lambda K_t \equiv N_{t+1}^s$. And, by claim 2(iii), we know that this group also hold all the money stock, M. The group's savings portfolio (N_{t+1}^s, M) satisfies (21), which can be simplified to:

$$\begin{aligned} & (1-\pi)\, E_t \left[\frac{(r_{t+1} + \lambda q_{t+1})/q_t - p_{t+1}/p_t}{(r_{t+1} + q_{t+1}\lambda) N_{t+1}^s + p_{t+1} M} \right] \\ = \; & \pi E_t \left[\frac{p_{t+1}/p_t - [r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q_{t+1}^R]/q_t}{[r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q_{t+1}^R] N_{t+1}^s + p_{t+1} M} \right]. \end{aligned} \tag{24}$$

Equation (24) lies at the heart of the model. When there is no investment opportunity at date t+1, so that the partial liquidity of equity doesn't matter, the return on equity, $(r_{t+1} + \lambda q_{t+1})/q_t$, exceeds the return on money, p_{t+1}/p_t : the LHS of (24) is positive. However, when there is an investment opportunity, the effective rate of return on equity, $[r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1 - \phi_{t+1})\lambda q_{t+1}^R]/q_t$, is less than the return on money: the RHS of (24) is positive. These return differentials have to be weighted by the respective probabilities and marginal utilities. Note that, because of the impact of idiosyncratic risk on the RHS, the liquidity premium of equity over money in the LHS may be substantial and may vary through time.

Aside from the liquidity shock ϕ_t and the technology parameter A_t which follow an exogenous stationary Markov process, the only state variable in this system is K_t , which evolves according to

$$K_{t+1} = \lambda K_t + I_t. \tag{25}$$

Restricting attention to a stationary price process, the competitive equilibrium can be defined recursively as a function (I_t, p_t, q_t, K_{t+1}) of the aggregate state (K_t, A_t, ϕ_t) that satisfies $(11), (22) - (25)$, together with the law of motion of A_t and ϕ_t.

From these equations it can be seen that there are rich interactions between quantities (I_t, K_{t+1}) and asset prices (p_t, q_t). In this sense, our economy is similar to Keynes (1936): (23) and (24) are akin to IS and LM equations.

In steady state, when $a_t = a$ (the RHS of (12) with $A_t = A$) and $\phi_t = \phi$, capital stock K, investment I, and prices p and q, satisfy $I = (1-\lambda)K$ and

$$\pi\beta r + \pi\beta l = \left[1 - \lambda + \pi\lambda\left(1-\beta\right)\frac{1-\phi}{1-\theta}\right]\left(1-\theta q\right) - \pi\beta\lambda\phi q \tag{26}$$

$$\begin{aligned} \beta r - (1-\beta)l &= \left[1 - \lambda + \pi\lambda\left(1-\beta\right)\frac{1-\phi}{1-\theta}\right] \\ &\quad + (1-\beta)\left(1 - \pi\frac{1-\phi}{1-\theta}\right)\lambda q \end{aligned} \tag{27}$$

$$r - (1-\lambda)q = \pi\lambda\frac{1-\phi}{1-\theta}\left(q-1\right)\frac{q + (l/\chi)}{r + \lambda\frac{1-\phi}{1-\theta} + \lambda\frac{\phi-\theta}{1-\theta}q + (l/\chi)}, \tag{28}$$

where $r = aK^{\alpha-1}$, $l = pM/K$, and $\chi \equiv \theta(1-\lambda) + (1-\pi+\pi\phi)\lambda$ (the steady-state fraction of equity held by non-investing entrepreneurs at the end of a period).

Equations (26), (27) and (28) can be viewed as a simultaneous system in three unknowns: the price of capital, q; the gross profit rate on capital, r; and the value of the money stock as a fraction of total capital, l. (26) and (27) can be solved for a r and l, each as affine functions of q, which when substituted into (28) yield a quadratic equation in q with a unique positive solution. Assumption 2 is sufficient to ensure that this solution lies strictly above 1 (but below $1/\theta$). We can also show that Assumption 2 is the necessary and sufficient condition for money to have value: $p > 0$.

As a prelude to the dynamic analysis that we undertake later on, notice that the technology parameter A only affects the steady-state system through the gross profit term $r = aK^{\alpha-1}$. That is, a rise in the steady state value of A increases the capital stock, K, but does not affect q, the price of capital. The price of money, p, increases to leave $l = pM/K$ unchanged.

It is interesting to compare our economy, in which the liquidity constraints (5) and (6) bind for investing entrepreneurs, to a "first-best" economy without such constraints. Consider steady states. In the first-best economy, the price of capital would equal its cost, 1; and the capital stock, K^* say, would equate the return on capital, $aK^{\alpha-1} + \lambda$, to the agents' common subjective return, $1/\beta$. (See Claim 1.) We show below that in our constrained economy, the level of activity – measured by the capital stock K – is strictly below K^*. Because of the partial liquidity of equity, the economy fails to transfer enough resources to the investing entrepreneurs to achieve the first-best level of investment.

On account of the liquidity constraint, there is a wedge between the marginal product of capital and the expected rate(s) of return on equity. It turns out that the expected rate(s) of return on equity and the rate of return on money all lie below the time preference. Intuitively, because the rates of return on assets to savers are below their time preference rate, they do not save enough to escape the liquidity constraint when they have an opportunity to invest in future.

Claim 3 *In the neighborhood of the steady state monetary economy,*

(i) the stock of capital, K_{t+1} is less than in the first-best (unconstrained) economy:

$$K_{t+1} < K^* \Leftrightarrow E_t\left(a_{t+1}K_{t+1}^{\alpha-1} + \lambda\right) > \frac{1}{\beta},$$

(ii) the expected rate of return on equity (assuming the saver does not have investment opportunity at date t+1) is lower the time preference rate:

$$E_t\frac{a_{t+1}K_{t+1}^{\alpha-1} + \lambda q_{t+1}}{q_t} < \frac{1}{\beta},$$

(iii) the expected rate of return on money is yet lower:

$$E_t\frac{p_{t+1}}{p_t} < E_t\frac{a_{t+1}K_{t+1}^{\alpha-1} + \lambda q_{t+1}}{q_t},$$

(iv) the expected rate of return on equity contingent on having an investment opportunity in the next period is lower still:

$$E_t\frac{a_{t+1}K_{t+1}^{\alpha-1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q_{t+1}^R}{q_t} < \frac{p_{t+1}}{p_t}.$$

Claim 3(iii) follows directly from (28), given that in steady state $q > 1$. This difference between the expected return on equity and money, reflecting the liquidity premium, equals the nominal interest rate on equity.[8]

In our monetary economy, there are a spectrum of interest rates. In descending order: the expected marginal product of capital, the time preference rate, the expected rate of return on equity, the expected rate of return on money, and the expected rate of return on equity contingent on the saver having an investment opportunity in the next period. Thus in our economy the impact of asset markets on aggregate production cannot be summarized by a single real interest rate as in some popular models such as Woodford (2003). Equally, it would be misleading to use the rates of returns on money or equity to calibrate the time preference rate.

The fact that the expected rates of return on equity and money are both lower than the time preference rate justifies our earlier assertion that workers will not choose to save by holding capital or money.[9] (Of course, if workers could borrow against their future labor income they would do so. But we have ruled this out.) In steady state, workers enjoy a constant consumption equal to their wages.

The reason why an entrepreneur saves, and workers do not, is because the entrepreneur is preparing for his next investment opportunity. And the entrepreneur saves using money as well as equity, despite money's particularly low return, because he anticipates that he will be liquidity constrained at the time of investment. Along a typical time path, he experiences episodes without investment, during which he consumes part of his saving. As the return on saving – on both equity and money – is less than his time preference

[8] By the Fisher equation, the nominal interest rate on equity equals the net real return on equity plus the inflation rate. But minus the inflation rate equals the net real return on money. Hence the nominal interest rate on equity equals the real return on equity minus the return on money, i.e. the liquidity premium. Because our money is broad money (all assets that are as liquid as fiat money), our nominal interest rate is akin to the interest rate in Keynes (1936): the difference in the rate of return on partially liquid assets versus that on fully liquid assets.

[9] Workers may save if they faced their own investment opportunity shocks. Suppose, for example, that each worker randomly faces a "health shock" which entails immediately spending some fixed amount ζ in order to maintain his human capital. (Health insurance may cover some of the cost, but the patient has to make a co-payment from his own pocket). Then, if the resaleability of equity is low, a worker may choose to save entirely in money enough to cover the amount ζ. The point is that even though the rate of return on equity is higher than money, on account of the resaleability constraint he would need to save more in equity than money, which may be less attractive given that the rate of return on equity is lower than his time preference rate. See Kiyotaki and Moore (2005a) for details.

rate, the value of his net worth gradually shrinks, as does his consumption. He only expands again at the time of investment. In the aggregate picture, we do not see all this fine grain. But it is important to realize that, even in steady state, the economy is made up of a myriad of such individual histories.

3 Dynamics and Numerical Examples

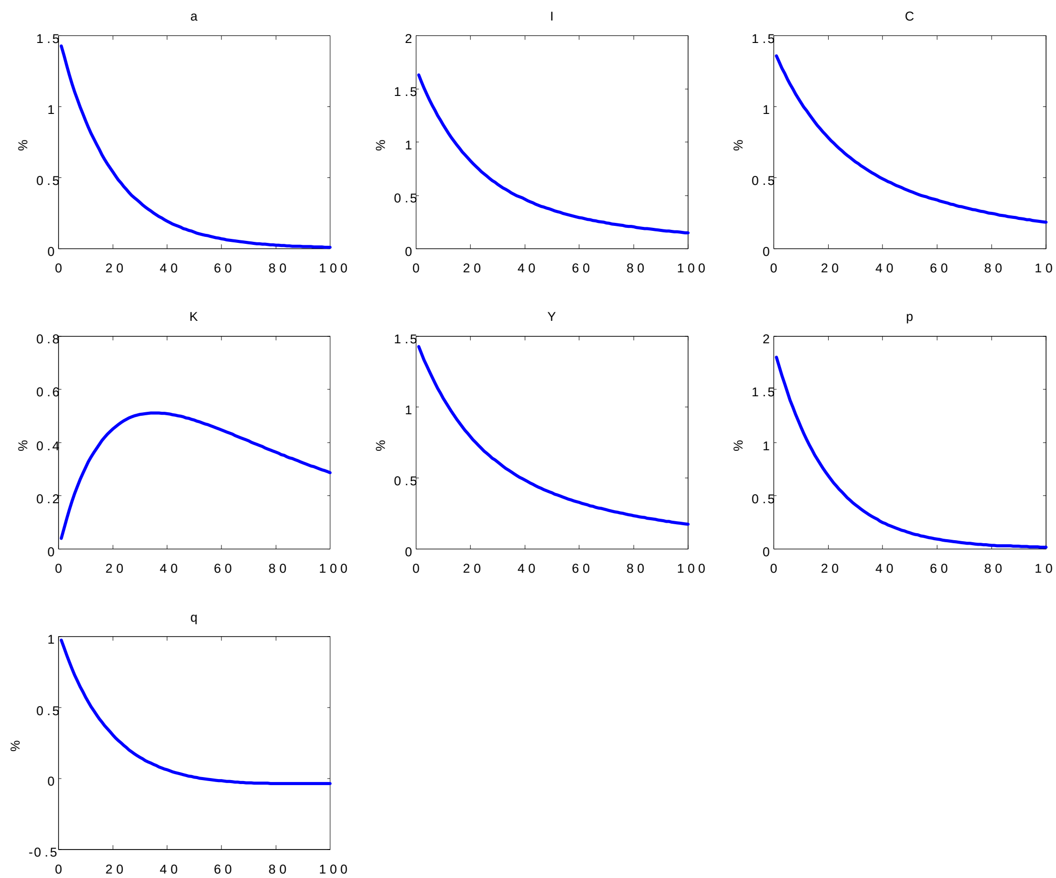
In order to examine the dynamics of our economy, let us present numerical examples by specifying a law of motion for productivity and liquidity (A_t, ϕ_t). Suppose that (A_t, ϕ_t) follows independent AR(1) processes so that $a_t = \gamma \left(\frac{1-\gamma}{\omega}\right)^{\frac{1-\gamma}{\gamma+\nu}} (A_t)^{\frac{1+\nu}{\gamma+\nu}}$ (in (12)) and ϕ_t follow AR(1) as

$$a_t - a = \rho_a (a_{t-1} - a) + \varepsilon_{at}, \tag{29}$$

$$\phi_t - \phi = \rho_\phi (\phi_{t-1} - \phi) + \varepsilon_{\phi t}, \tag{30}$$

where ρ_a and $\rho_\phi \in (0, 1)$ and we set $\rho_a = \rho_\phi = 0.95$ for calibration. The variables ε_{at} and $\varepsilon_{\phi t}$ are iid innovations of the levels of productivity and liquidity, which have mean zero and are mutually independent. We present our numerical examples to illustrate mainly qualitative features of our model. We follow Del Negro et. al. (2011) for choosing parameters. In particular, we consider one period is quarterly and use $\pi = 0.05$ (arrival rate of investment opportunity), $\theta = 0.19$ (mortgageable fraction of new investment), $\phi = 0.19$ (resaleable fraction of equity in the steady state), $\gamma = 0.4$ (share of capital), $\nu = 1$ (inverse of Frisch elasticity of labor supply), $\beta = 0.99$ (utility discount factor), $\lambda = 0.975$ (one minus depreciation rate). $Figure$ 1 shows the impulse response function to a 1% increase in A_t, which increases a_t by $\frac{1+\nu}{\gamma+\nu} = 1.43\%$:

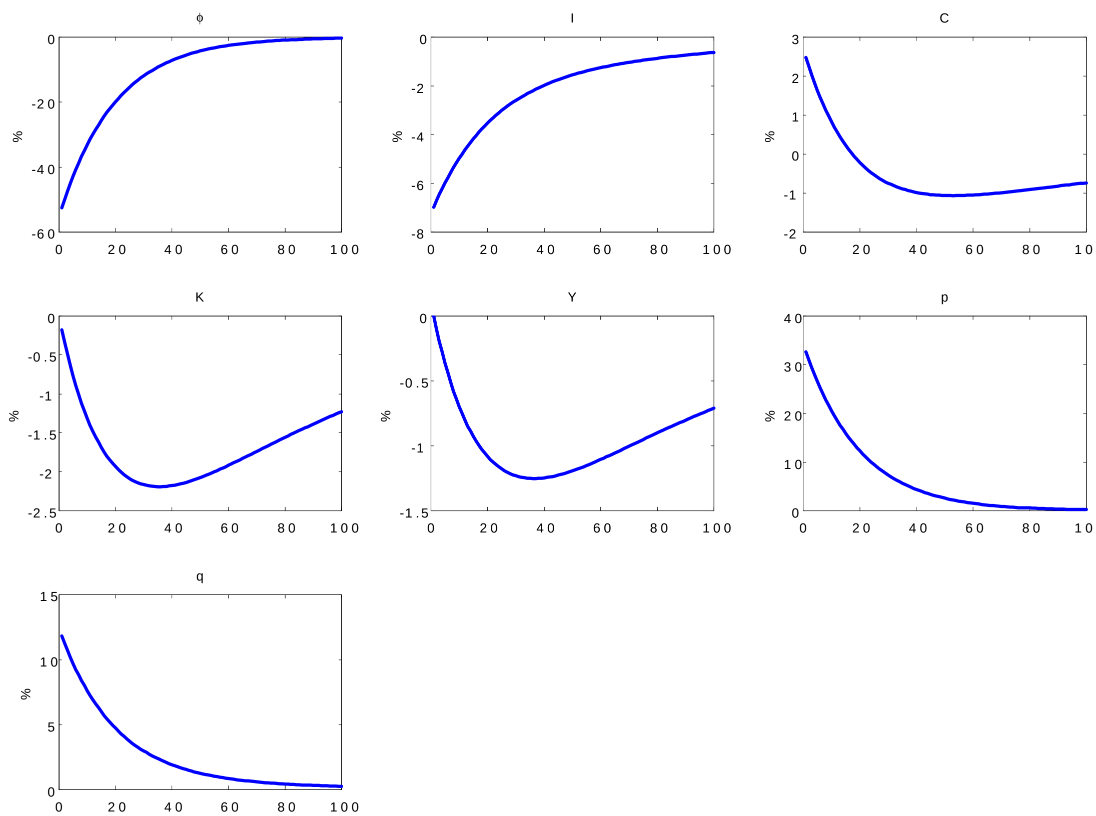
Figure 1. Impulse Responses of Basic Economy to Productivity Shock



Because capital stock is pre-determined, and the labor market clears, output increases by 1.43% (the same proportion as a_t). Then from the goods market equilibrium condition (23) , we observe the asset prices (p_t, q_t) have to increase together with productivity in order to increase consumption and investment in line with the larger output. Although investment is more sensitive to the asset prices and thus increases proportionately more than consumption, the aggregate consumption of both entrepreneurs and workers increase substantially (especially because workers' consumption is equal to their wage income). This is different from the first best allocation under *Condition* 1, in which consumption would be much smoother than investment because, without the binding liquidity constraints, consumption would depend upon permanent rather than current income. The co-movement of quantities and asset prices is also a unique features of the monetary equilibrium with binding liquidity constraints. In contrast, in the first best Tobin's q would always equal 1 and the value of money would always be zero.

Now let us consider liquidity shocks. $Figure$ 2 shows the impulse response of quantities and asset prices against a 50% fall in the resaleability of the equity.

Figure 2. Impulse Responses of Basic Economy to Liquidity Shock



When the resaleability of equity falls, and only slowly recovers, the investing entrepreneurs are less able to finance downpayment from selling their equity holdings, and so investment decreases substantially. Capital stock and output gradually decrease with persistently lower investment. Also the entrepreneurs without investment opportunities now find money more attractive than equity as a means of saving (holding the rates of return unchanged), given that they can resale a smaller fraction of their equity holding when future investment opportunities arise (see (24)). Thus, the value of money increases compared to the equity price in order to restore asset market equilibrium. This can be thought of as a "flight to liquidity": a flight from equity to money.

Notwithstanding this flight from equity, the equity price tends to rise with the fall in the liquidity. One way to understand why is to think of the gap between Tobin's q and unity as a measure of the tightness of the liquidity constraint, which increases when the resaleability of equity falls. Another way is to observe that, because output is not affected initially (given full employment), consumption must increase to maintain equilibrium in the goods market; and consumption rises through the wealth effect of a rise in asset prices. This negative co-movement between investment, asset prices and consumption is a shortcoming of our basic model – a shortcoming shared by many macroeconomic model with flexible prices.[10] We address this in the next section.

Note that, in contrast to our monetary equilibrium, the first best allocation would not react to the liquidity shock as the liquidity constraint would not be binding.

4 Full Model with Storage and Government

We now present the full model. The negative co-movement in the basic model between investment, asset prices and consumption can be reversed by expanding the model to include an alternative liquid means of saving: storage. Specifically, suppose that an agent can store $\sigma_t z_{t+1}$ units of goods at date t to obtain z_{t+1} units of goods at date t+1, where z_{t+1} must be nonnegative. Although the storage technology has constant returns to scale at the individual level, it is decreasing returns to scale at the society level: σ_t is an increasing function of the aggregate quantity of storage Z_{t+1},

[10] Shi (2011) points out that in our basic model it is difficult for a liquidity shock to generate a positive co-movement in aggregate investment and the price of equity.

$$\sigma_t = \sigma\left(Z_{t+1}\right) = \left(\frac{Z_{t+1}}{\zeta_0}\right)^{\zeta}, \text{ where } \zeta_0,\ \zeta > 0.$$

Storage represents all the various means of short-term saving besides money, such as consumer durables or net foreign asset (domestic residents can save in foreign assets but cannot borrow from foreigners).

To complete the model, we also introduce government. Our goal here is simply to explore the effects on equilibrium of an exogenous government policy rule. We make no attempt to explain government behavior. At the start of date t, suppose the government holds N_t^g equity. Unlike entrepreneurs, the government cannot produce new capital. However, it can engage in open market operations, to buy (sell) equity by issuing (taking in) money – it has sole access to a costless money-printing technology. Any sale of equity is subject to the same constraint as (5).[11] Finally, the government can purchase goods, or transfer goods to the workers (a negative would correspond to a lump-sum tax of the workers). Let G_t denote the total government purchases. Assume that G_t does not affect the entrepreneurs, which leaves intact our analysis of their behavior. We assume that N_t^g and G_t are not so large that the private economy switches regimes. That is, we are still in an equilibrium where the liquidity constraints bind for investing entrepreneurs and money is valuable.

If M_t is the stock of money privately held by entrepreneurs at the start of date t, then the government's flow-of-funds constraint is given by

$$G_t + q_t\left(N_{t+1}^g - \lambda N_t^g\right) = r_t N_t^g + p_t(M_{t+1} - M_t) = r_t N_t^g + \left(\mu_t - 1\right) L_t, \quad (31)$$

where $L_t \equiv p_t M_t$ are real balances, and $\mu_t \equiv \frac{M_{t+1}}{M_t}$ is the money supply growth rate. That is, cost of the government's purchases of output and equity must be met by the dividends from its equity holding plus seigniorage revenues. Since government is a large agent, at least relative to each of the private agents, open market operations will affect the prices p_t and q_t.

We will suppose that the government follows a rule for its open market operations and fiscal policy:

$$\frac{N_{t+1}^g}{K} = \psi_a \frac{a_t - a}{a} + \psi_\phi \frac{\phi_t - \phi}{\phi} \quad (32)$$

[11] The government also is subject to the same resaleability constraint as the entrepreneur: $N_{t+1}^g \geq (1 - \phi_t)\lambda N_t^g$.

$$G_t = \xi\left[(r_t + \lambda q_t)N_t^g - (L_t - L)\right], \tag{33}$$

where ψ_a, ψ_ϕ and ξ are policy parameters, and K and L are capital stock and real money balances in the non-stochastic steady state. The first equation is government's feedback rule for its open market operations: it chooses the size of open market operation (ratio of government equity holding to total equity supply in the steady state) as a function of the proportional deviations of productivity and liquidity from the steady state levels. This rule implies that the government's equity holding is zero in the steady state. The second equation is the fiscal policy rule: the government adjusts its goods purchases to be proportional to the deviation of its net asset holdings from steady state at the beginning of each period. To limit the length of our discussion, we will here report only on simulations with $\xi = 0$ so that $G_t = 0$; i.e. where all the fiscal adjustment is done through the money supply growth rate μ_t.

The earlier analysis carries through, with obvious modifications. The total supply of equity (which by construction is equal to the aggregate capital stock) equals the sum of the government's holding and the aggregate holding of the entrepreneurs (denoted by N_{t+1}):

$$K_{t+1} = N_{t+1}^g + N_{t+1}. \tag{34}$$

Workers consume all their disposable income, and, given the form of their preferences in (8), government policy does not affect their labor supply. Equations (22), (23) and (24) are modified to:

$$(1 - \theta q_t)\, I_t = \pi\left\{\beta\left[(r_t + \lambda\phi_t q_t)N_t + L_t + Z_t\right] - (1-\beta)(1-\phi_t)\lambda q_t^R N_t\right\} \tag{35}$$

$$\begin{aligned} a_t K_t^\alpha + Z_t &= I_t + \sigma\left(Z_{t+1}\right) Z_{t+1} + G_t + (1-\beta)\cdot \\ &\quad \left\{[r_t + (1 - \pi + \pi\phi_t)\lambda q_t + \pi\left(1-\phi_t\right)\lambda q_t^R]N_t + L_t + Z_t\right\} \end{aligned} \tag{36}$$

$$\begin{aligned} &(1-\pi)\, E_t\left[\frac{(r_{t+1} + \lambda q_{t+1})/q_t - L_{t+1}/(\mu_t L_t)}{(r_{t+1} + q_{t+1}\lambda)N_{t+1}^s + L_{t+1} + Z_{t+1}}\right] \\ &= \pi E_t\left[\frac{L_{t+1}/(\mu_t L_t) - [r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q_{t+1}^R]/q_t}{[r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q_{t+1}^R]N_{t+1}^s + L_{t+1} + Z_{t+1}}\right] \end{aligned} \tag{37}$$

$$(1-\pi)\, E_t\left[\frac{(r_{t+1} + \lambda q_{t+1})/q_t - (1/\sigma(Z_{t+1}))}{(r_{t+1} + q_{t+1}\lambda)N_{t+1}^s + L_{t+1} + Z_{t+1}}\right]$$

$$= \pi E_t \left[\frac{(1/\sigma(Z_{t+1})) - [r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q^R_{t+1}]/q_t}{[r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1-\phi_{t+1})\lambda q^R_{t+1}]N^s_{t+1} + L_{t+1} + Z_{t+1}} \right], \qquad (38)$$

where $N^s_{t+1} = \theta I_t + \phi_t \pi \lambda N_t + (1-\pi)\lambda N_t + \lambda N^g_t - N^g_{t+1}$. In the investment equation, (35), entrepreneurs use their money, storage and the resaleable portion of their equity – net of their consumption – to finance the downpayment. In the goods market equilibrium, (36) , output (net of the worker's consumption) plus storage return equals the sum of investment, new storage, government purchases and entrepreneurs' consumption. (If storage were considered a net foreign asset, then the accumulation of net foreign assets, $Z_{t+1} - Z_t$, would be the current account.) The portfolio equation (37) gives the trade-off between holding equity and money. And the new portfolio equation (38) gives the trade-off between holding equity and storage.

Restricting attention to a stable price process, competitive equilibrium is defined recursively as functions $\left(I_t, r_t, L_t, q_t, Z_{t+1}, K_{t+1}, N_{t+1}, N^g_{t+1}, G_t, \mu_{t+1}\right)$ of the aggregate state $(K_t, Z_t, N^g_t, a_t, \phi_t)$ that satisfy (11),(25),(31) – (38) together with the exogenous law of motion of (a_t, ϕ_t) .[12]

How does the presence of storage as alternative means of liquid saving storage alter the impulse responses? $Figure$ 3 compares the impulse responses to a liquidity shock in the model without storage (taken from $Figure$ 2) and in the model with storage. We choose a storage technology that has very close to constant returns to scale ($\zeta = 0.0001$), and is such that the steady-state level of storage ($\zeta_0 = 0.5$) is modest compared to the steady-state capital stock ($K = 9.49$).

In response to the fall in the resaleability of equity, storage increases sharply, and investment falls more significantly than the economy without storage, leading to a more significant fall in output.[13] Importantly, consumption can now also fall along with investment, as output is soaked up by the sharp rise in storage.

Also, because money and storage are very close substitutes – with a rate

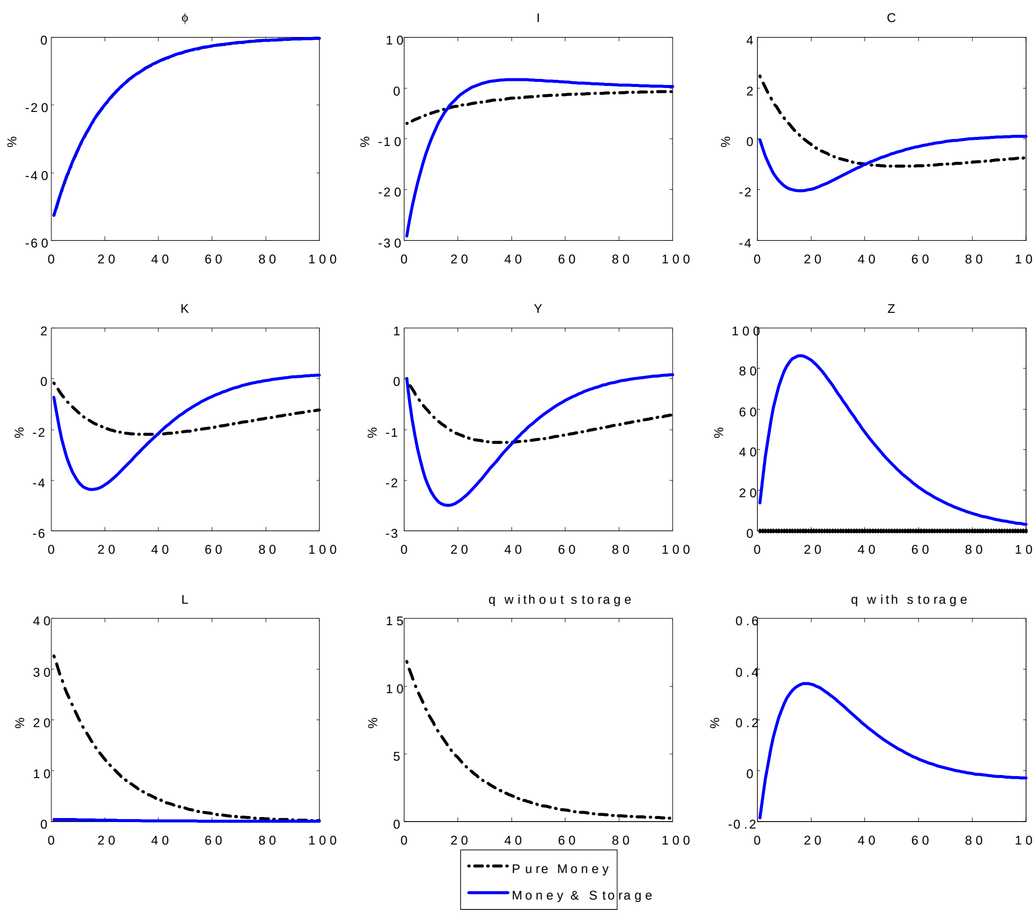
[12] If there were a lump-sum transfer of money to the entrepreneurs (a helicopter drop), then aggregate quantities would not change in our economy given that prices and wages are flexible. The consumption and investment of individual entrepreneurs would be affected, however, because there would be some redistribution.

[13] In Bernanke and Gertler (1989), this is called "disintermediation": with greater friction in financial markets, more funds bypass those markets and are instead channelled directly into alternative investments – here, into storage – that may be productively inferior.

of return very close to one – the price of money is stable. Real balances hardly increase. As a consequence, the "flight to liquidity" induces the equity price to fall a little, at least initially.

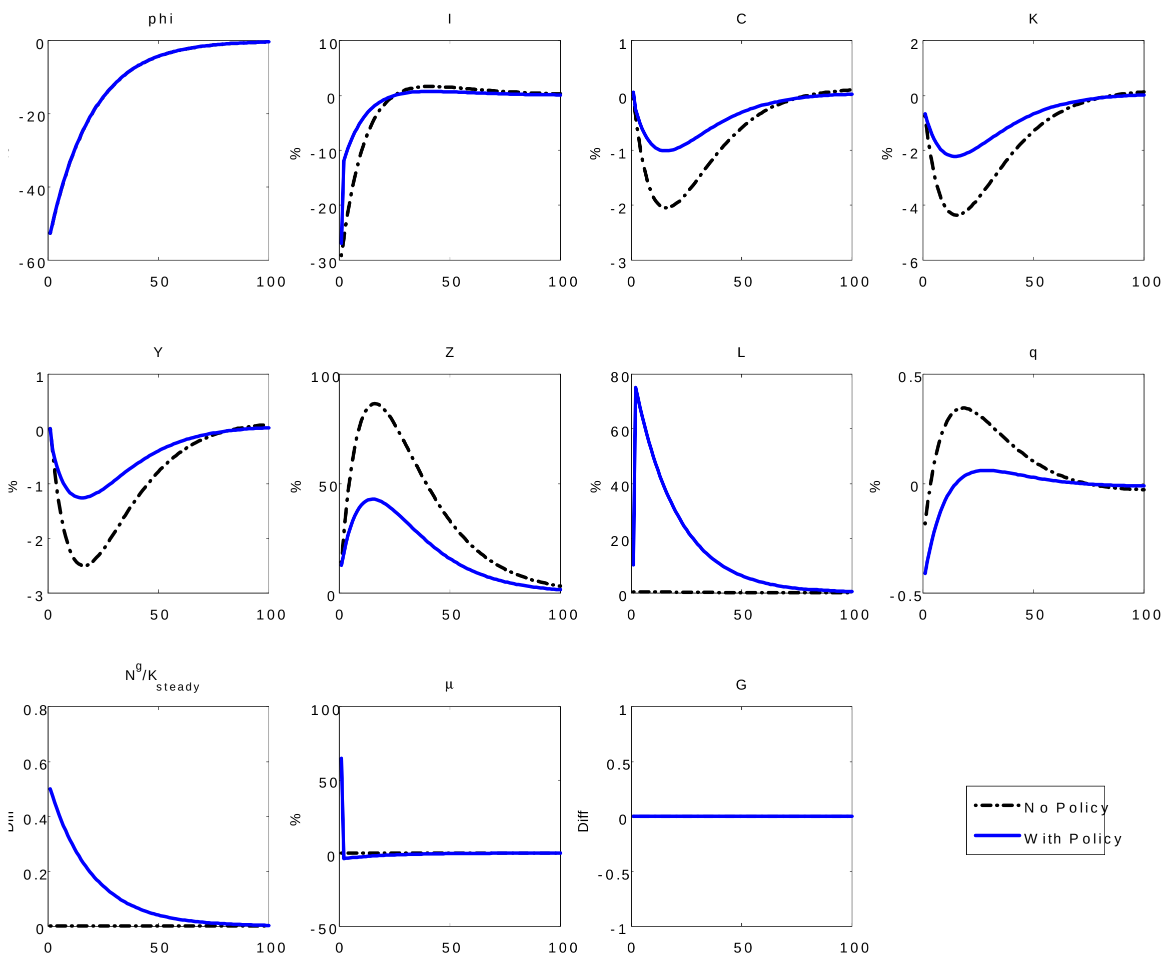
Taking these findings together, we see that the presence of an alternative liquid means of saving has overcome the shortcomings of our basic model. Quantities (investment and consumption) and asset prices move together, as storage serves as a buffer stock to absorb output and stabilize the value of money.

Figure 3. Impulse Responses of Economy with Storage to Liquidity Shock



How might the government, through its central bank, conduct open market operations in response to the liquidity shock? A first-best allocation would not be affected by a liquidity shock. With this benchmark in mind, in our monetary economy the central bank should use open market operations to offset the effects of the liquidity shock, by setting the feedback rule coefficient ψ_ϕ to be negative in (32). That is, the central bank should counteract the negative shock by purchasing equity with money, in order to – at least partially – restore the liquidity of investing entrepreneurs. *Figure* 4 compares the impulse responses of the economy without this policy rule ($\psi_\phi = 0$ and $\xi = 0$, taken from *Figure* 3) and with the rule ($\psi_\phi = -0.1$).

Figure 4. Impulse Responses of Full Model to Liquidity Shock

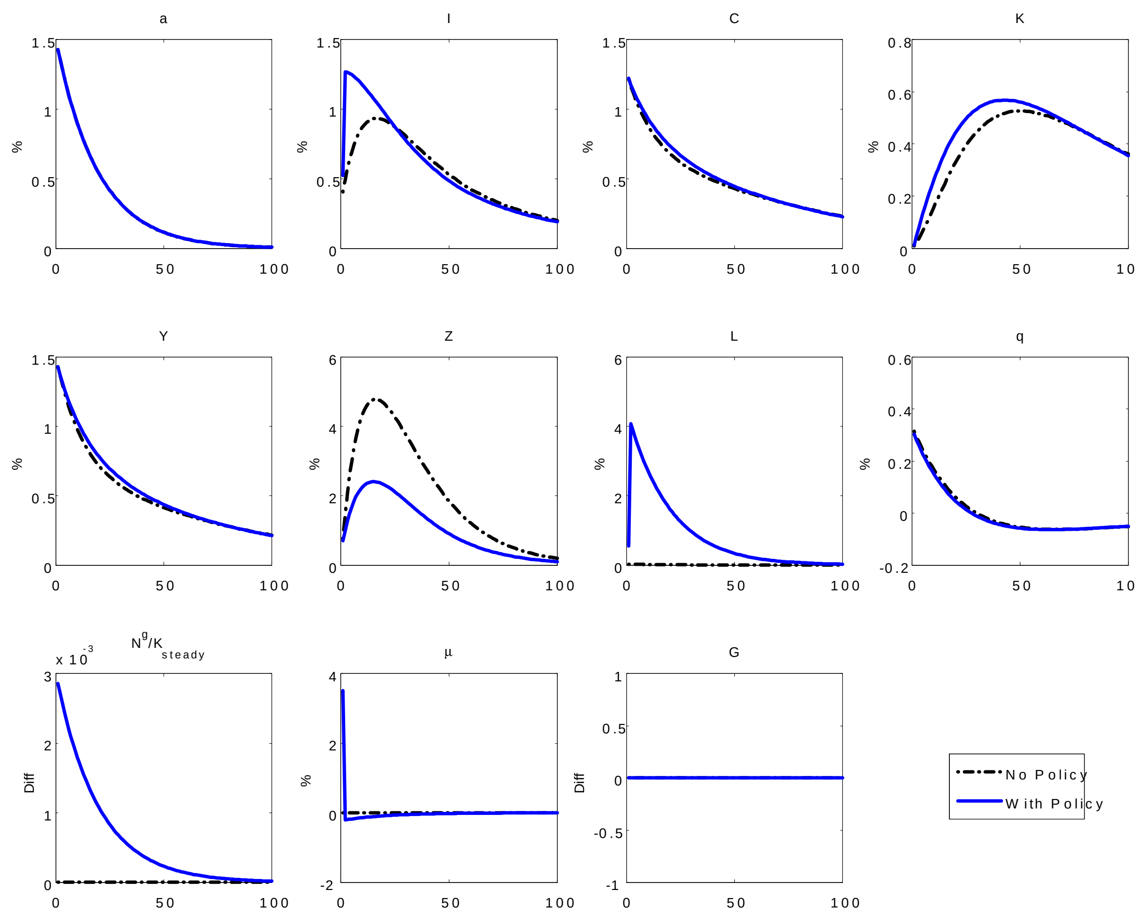


The central bank's purchases of equity with money causes real balance to increase sharply, notwithstanding the relatively stable price of money. Storage increases less than in the economy without the policy intervention. Investment falls initially by 30% – almost as much as in the case of no policy, because at the time of the shock the investing entrepreneurs' portfolios are predetermined. However, in the following period, investing entrepreneurs (most of whom were savers in previous period) have a larger proportion of liquid assets thanks to the policy intervention, and investment recovers to a level of 10% below the steady state. Thus capital stock and output do not fall as much as in the economy without intervention.

After the initial purchase of equity, government runs a surplus because equity yields a higher return. It uses this surplus to reduce the money supply by setting $\mu_t < 1$ (assuming no adjustment to government purchases). Because this deflationary policy rewards money holders, the flight to liquidity is more pronounced: the equity price falls as a result.

In contrast, how might the central bank use open market operations in response to a productivity shock? Once more taking the first-best allocation as a benchmark, the problem of our laissez-faire monetary economy is that investment does not react enough to productivity shocks and consumption is not smooth enough. Here the central bank should provide liquidity procyclically to accommodate productivity shocks, by setting the feedback coefficient of ψ_a to be positive in (32). $Figure$ 5 compares the impulse response functions of the laissez-faire monetary economy with an accommodating monetary policy ($\psi_a = 0.2$ and $\xi = 0$). As productivity rises by 1.43%, the central bank buys equity with money to provide an additional 4% liquidity (again, notwithstanding the relatively stable price of money). Entrepreneurs hold more money and less illiquid equity, and thus investment more. Investment increases by 1.3% in the periods immediately following the shock, rather than increasing gradually as in the economy without the intervention. But whereas investment, and hence capital stock and output, all increase more because of the policy, storage and consumption increase less.

Figure 5, Impulse Responses of the Full Model to Productivity Shock



The efficacy of these open market operations relies on the purchase of an asset – here, equity – which is only partially resaleable and hence earns a non-trivial liquidity premium. If the liquidity premium of short-term government bonds is very low (as in Japan since the late 1990s), then traditional open market operations will only serve to change the composition of broad money and will have limited effects. The recent unorthodox policy of the Federal Reserve Bank (and the Bank of England), such as the Term Security Lending Facility, is an attempt to increase liquidity by supplying treasury bills against only partially resaleable securities, such as mortgage backed securities.

5 Related Literature and Final Remarks

We hope to have succeeded in constructing a model of money and liquidity in the tradition of Keynes (1936) and Tobin (1969). The two key equations of our model, (23) and (24) – which are generalized in (36) , (37) and (38) – have the flavor of the Keynesian IS-LM system. We follow Tobin in placing emphasis on the spectrum of liquidity across different classes of asset. Also, Tobin's q-theory finds echo in our model through the central role played by the equity price q: driving the feedback from asset markets to the rest of the economy. Our policy prescriptions – use open market operations to change the liquidity mix of the private sector's asset holdings – parallel those in Metzler (1951). Perhaps, with its focus on liquidity, our framework harks back to an earlier tradition of interpreting Keynes, and has less in common with the formal Keynesian literature, with its emphasis on sticky prices, that has been dominant in the past few decades.

This paper is part of the recent literature on macroeconomics with financial frictions, that includes Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Holmstrom and Tirole (1998).[14] Naturally, the common thread of this literature has been some form of borrowing constraint, akin to our θ-constraint. Our innovation here is to combine it with the ϕ-constraint, the resaleability constraint. We have shown that the presence of these two constraints opens up the possibility for fiat money to circulate, to lubricate the transfer of goods from savers to investors. There is a wedge between money and other assets, that arises out of the assumed difference in their resaleability.

Wedges between assets can be generated in other ways. In limited participation models, agents may have different access to asset markets.[15] Models with spatially separated markets – island models – assume that agents cannot visit all markets within the period, which limits trade across assets. Some models combine geographical separation with asynchronization, where agents have access to asset markets at different times.[16] If the assumption of competitive markets is dropped, as in matching models, assets can ex-

[14] Surveys can be found in Bernanke, Gertler and Gilchrist (1999), Gertler and Kiyotaki (2010), Brunnermeier, Eisenbach and Sannikov (2011).

[15] See, for example, Allen and Gale (1994, 2007).

[16] See, for example, Townsend (1987), Townsend and Wallace (1987), Freeman (1996a, 1996b), and Green (1999).

hibit different degrees of resaleability.[17] And there is a long tradition in the banking and finance literature that, implicitly or explicitly, has to do with the limited resaleability of securities, dating back at least to Diamond and Dybvig (1983).[18]

We should end by stressing that if, in particular, our model is to be used for proper policy analysis then considerably more research is still needed. While it might be argued that our θ-ϕ framework has the virtue of simplicity, as they stand the borrowing and resaleability constraints are too stylized in nature, too reduced-form. The borrowing constraint can be rationalized by invoking a moral hazard argument, viz., to produce future output from new capital requires the specific skill of the investing entrepreneur, and he can renege on his promises. But the resaleability constraint requires more modelling, not least because we need to understand where the liquidity shocks, the shocks to ϕ, come from.[19] Can policies be devised that directly dampen these shocks (or even raise the average value of ϕ), rather than merely dealing with their effects?

To analyze the effects of open market operations over the business cycle, we assumed that the government can commit to a policy. But can it? This question calls for further modelling too, because if the government could commit to, say, a deflationary monetary policy that followed the Friedman rule (set the real return on money to equal agents' subjective discount rate), then it would in effect be using its taxation powers to substitute perfect public commitment for imperfect private commitment. In the long run, can the government be trusted more than the private sector? And to what extent do future tax liabilities crowd out a private agent's ability to issue credible promises to others?[20] These thorny issues warrant much careful thinking.

[17] Matching models that can be easily used for policy analysis include Shi (1997), Lagos and Wright (2005), and Nosal and Rocheteau (2011).

[18] For attempts to incorporate banking into standard business cycle models, see, for example, Wiiliamson (1987), Gertler and Kiyotaki (2010), Gertler and Karadi (2011).

[19] Kiyotaki and Moore (2003) shows how the resaleability constraint can arise endogenously due to adverse selection and how securitization may mitigate the adverse selection.

Other macroeconomic models of adverse selection in asset markets inclde Eisfeldt (2004), Moore (2010), Kurlat (2011).

[20] A related question would be: If the government has a superior power to force private agents to pay, why doesn't it provide them with finance directly?

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7 Appendix

7.1 Proof of Claim 1

We construct a competitive equilibrium which satisfies Claim 1 under Condition 1. Suppose that inequalities (5), (6) and (10) are not binding. Then from (7), we need $q_t = 1$. (If q_t were strictly larger than one, investing entrepreneurs would invest arbitrary large amount and (5) would be binding. If q_t were strictly smaller than one, investing entrepreneurs would not invest at all, which is not consistent with the equilibrium with positive gross investment in the neighborhood of the steady state). Then the choice of entrepreneurs and workers in the competitive equilibrium imply:

$$\begin{aligned} 1 &= E_t\left[\beta\frac{c_t}{c_{t+1}}\left(r_{t+1}+\lambda\right)\right] = E_t\left[\beta\frac{c'_t}{c'_{t+1}}\left(r_{t+1}+\lambda\right)\right], \quad \text{(A1)} \\ r_t &= \gamma\left(\frac{\ell_t}{K_t}\right)^{1-\gamma} \\ w_t &= (1-\gamma)\left(\frac{K_t}{\ell_t}\right)^{\gamma} = \omega(\ell_t)^{\nu} \\ A_t(K_t)^{\gamma}(\ell_t)^{1-\gamma} &= C_t + c'_t + K_{t+1} - \lambda K_t, \end{aligned}$$

where K_t is aggregate capital stock, ℓ_t is aggregate labor, and C_t and c'_t are aggregate consumption of entrepreneurs and workers. Because these are the conditions for the first best allocation, the competitive equilibrium achieves the first best allocation if (5), (6) and (10) are not binding. In this competitive equilibrium, we get $p_t = 0$. (If p_t were strictly positive with non-binding (6), we would have $1 = E_t\left(\beta\frac{c_t}{c_{t+1}}\frac{p_{t+1}}{p_t}\right)$ which would not be satisfied in the neighborhood of the steady state).

Consider this competitive economy in which workers do not save so that $c'_t = w_t\ell_t$ and aggregate investment is equal to aggregate saving of entrepreneurs:

$$I_t = K_{t+1} - \lambda K_t = r_t K_t - C_t.$$

Using (7) with $q_t = 1$ and $p_t = 0$, the inequality (5) for the investing entrepreneur becomes

$$r_t n_t - c_t = n_{t+1} - \lambda n_t \geq (1-\theta)i_t - \phi_t \lambda n_t.$$

Aggregating this inequality for all the investing entrepreneurs, observing the arrival of the investment opportunity is iid. across entrepreneurs and over time, we have

$$\pi\left(r_t K_t - C_t\right) = \pi I_t \geq (1-\theta) I_t - \phi_t \lambda \pi K_t. \tag{A2}$$

In the steady state, Condition 1 implies

$$\pi(1-\lambda)K > (1-\theta)(1-\lambda)K - \phi\lambda\pi K.$$

Then in the neighborhood of the steady state (in which $I = (1-\lambda)K$), inequality ($A2$) is satisfied. Because $p_t = 0$, (6) and the second inequality of (10) are not binding. Therefore under Condition 1, we can find a competitive equilibrium in which the inequalities (5), (6) and (10) are not binding and $c_t' \simeq w_t \ell_t$ and $n_t' \simeq 0$ in the neighborhood of the steady state. Claim 1(iv) follows from ($A1$). *QED*.

7.2 Derivation of Consumption and Portfolio Equations

Let $V_t(m_t, n_t)$ be the value function of the entrepreneur who holds money and equity (m_t, n_t) at the beginning of the period t before meeting an opportunity to invest with probability π. The Bellman equation can be written as

$$\begin{aligned} V_t(m_t, n_t) &= \pi \cdot \underset{\substack{c_t^i, i_t, m_{t+1}^i, n_{t+1}^i \\ st.(5)(6)(7)}}{Max} \left\{\ln c_t^i + \beta E_t \left[V_t(m_{t+1}^i, n_{t+1}^i)\right]\right\} \\ &\quad + (1-\pi) \cdot \underset{\substack{c_t^s, m_{t+1}^s, n_{t+1}^s \\ st.(5)(6)(7),\ i_t = 0}}{Max} \left\{\ln c_t^s + \beta E_t \left[V_t(m_{t+1}^s, n_{t+1}^s)\right]\right\}. \end{aligned}$$

Solving the flow-of-funds condition (14) and (15) for consumption c_t^s and c_t^i, the Bellman equation is

$$\begin{aligned} & V_t(m_t, n_t) \\ = \ & \pi \cdot \underset{m_{t+1}^i, n_{t+1}^i}{Max} \left\{ \begin{array}{c} \ln\left([r_t + \phi_t \lambda q_t + (1-\phi_t)\lambda q_t^R] n_t + p_t m_t - q_t^R n_{t+1}^i - p_t m_{t+1}^i\right) \\ +\beta E_t \left[V_t(m_{t+1}^i, n_{t+1}^i)\right] \end{array} \right\} \\ & + (1-\pi) \cdot \underset{m_{t+1}^s, n_{t+1}^s}{Max} \left\{ \begin{array}{c} \ln\left(\left(r_t + \lambda q_t\right) n_t + p_t m_t - q_t n_{t+1}^s - p_t m_{t+1}^s\right) \\ +\beta E_t \left[V_t(m_{t+1}^s, n_{t+1}^s)\right] \end{array} \right\}. \end{aligned}$$

Let R_{mt+1} be the rate of return on money from date t to date t+1 and let $R^{hh'}_{t+1}$ be the implied rate of returns on equity for the entrepreneur when her type is h ($h = i$ for investing and $h = s$ for saving or non-investing) at date t and h' at date t+1, i.e.,

$$\begin{aligned}
R_{mt+1} &= \frac{p_{t+1}}{p_t}, \\
R^{ss}_{t+1} &= \frac{r_{t+1} + \lambda q_{t+1}}{q_t},\ R^{si}_{t+1} = \frac{r_{t+1} + \lambda \phi_{t+1} q_{t+1} + \lambda \left(1 - \phi_{t+1}\right) q^R_{t+1}}{q_t}, \\
R^{is}_{t+1} &= \frac{r_{t+1} + \lambda q_{t+1}}{q^R_t},\ R^{ii}_{t+1} = \frac{r_{t+1} + \lambda \phi_{t+1} q_{t+1} + \lambda \left(1 - \phi_{t+1}\right) q^R_{t+1}}{q^R_t}.
\end{aligned}$$

Then the first order conditions are

$$1 = E_t \left[\pi \frac{\beta c^i_t}{c^{ii}_{t+1}} R^{ii}_{t+1} + (1 - \pi) \frac{\beta c^i_t}{c^{is}_{t+1}} R^{is}_{t+1} \right] \tag{A3}$$

$$1 > E_t \left[\pi \frac{\beta c^i_t}{c^{ii}_{t+1}} R_{mt+1} + (1 - \pi) \frac{\beta c^i_t}{c^{is}_{t+1}} R_{mt+1} \right] \tag{A4}$$

$$1 = E_t \left[\pi \frac{\beta c^s_t}{c^{si}_{t+1}} R^{si}_{t+1} + (1 - \pi) \frac{\beta c^s_t}{c^{ss}_{t+1}} R^{ss}_{t+1} \right] \tag{A5}$$

$$1 = E_t \left[\pi \frac{\beta c^s_t}{c^{si}_{t+1}} R_{mt+1} + (1 - \pi) \frac{\beta c^s_t}{c^{ss}_{t+1}} R_{mt+1} \right] \tag{A6}$$

where c^h_t is date-t consumption of the entrepreneur of type h and $c^{hh'}_{t+1}$ is date t+1 consumption of entrepreneur when her type is h at date t and h' at date t+1.

We guess that

$$\begin{aligned}
c^i_t &= (1 - \beta) \left\{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q^R_t] n_t + p_t m_t \right\}, \\
c^s_t &= (1 - \beta) \left\{ (r_t + \lambda q_t) n_t + p_t m_t \right\}, \\
n^i_{t+1} &= \beta \left\{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q^R_t] n_t + p_t m_t \right\} / q^R_t, \\
m^i_{t+1} &= 0, \\
n^s_{t+1} &= \beta f_t \left\{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q^R_t] n_t + p_t m_t \right\} / q_t, \\
m^s_{t+1} &= \beta (1 - f_t) \left\{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q^R_t] n_t + p_t m_t \right\} / p_t
\end{aligned}$$

$$\begin{aligned}
c^{ii}_{t+1} &= (1-\beta)\,[r_{t+1}+\phi_{t+1}\lambda q_{t+1}+(1-\phi_{t+1})q^R_t]n^i_{t+1}, \\
c^{is}_{t+1} &= (1-\beta)\,(r_{t+1}+\lambda q_{t+1})n^i_{t+1}, \\
c^{si}_{t+1} &= (1-\beta)\left\{[r_{t+1}+\phi_{t+1}\lambda q_{t+1}+(1-\phi_{t+1})q^R_t]n^s_{t+1}+p_{t+1}m^s_{t+1}\right\}, \\
c^{ss}_{t+1} &= (1-\beta)\left\{(r_{t+1}+\lambda q_{t+1})n^s_{t+1}+p_{t+1}m^s_{t+1}\right\}.
\end{aligned}$$

where $f_t \equiv q_t n^s_{t+1}/(p_t m^s_{t+1}+q_t n^s_{t+1}) \in (0,1)$ is the share of equity in portfolio of the non-investing entrepreneur. Under this guess we learn

$$\frac{c^{ii}_{t+1}}{c^i_t} = \beta R^{ii}_{t+1}, \text{ and } \frac{c^{is}_{t+1}}{c^i_t} = \beta R^{is}_{t+1},$$

and thus $(A3)$ is always satisfied. $(A4)$ can be written as

$$1 > E_t\left[\pi\frac{R_{mt+1}}{R^{ii}_{t+1}}+(1-\pi)\frac{R_{mt+1}}{R^{is}_{t+1}}\right]. \tag{A7}$$

We also learn that

$$\begin{aligned}
\frac{c^{si}_{t+1}}{c^s_t} &= \beta\left[f_t R^{si}_{t+1}+(1-f_t)R_{mt+1}\right], \\
\frac{c^{ss}_{t+1}}{c^s_t} &= \beta\left[f_t R^{ss}_{t+1}+(1-f_t)R_{mt+1}\right].
\end{aligned}$$

Thus $(A5)$ and $(A6)$ become

$$\begin{aligned}
1 &= E_t\left[\pi\frac{R^{si}_{t+1}}{f_t R^{si}_{t+1}+(1-f_t)R_{mt+1}}+(1-\pi)\frac{R^{ss}_{t+1}}{f_t R^{ss}_{t+1}+(1-f_t)R_{mt+1}}\right], \quad \text{(A8)} \\
1 &= E_t\left[\pi\frac{R_{mt+1}}{f_t R^{si}_{t+1}+(1-f_t)R_{mt+1}}+(1-\pi)\frac{R_{mt+1}}{f_t R^{ss}_{t+1}+(1-f_t)R_{mt+1}}\right] \quad \text{(A9)}
\end{aligned}$$

Then f_t times the RHS of $(A8)$ plus $1-f_t$ times RHS of $(A9)$ is always equal to 1, and one of $(A8)$ and $(A9)$ is not independent. Subtracting $(A9)$ from $(A8)$ side by side and arranging, we get

$$\pi E_t\left[\frac{R_{mt+1}-R^{si}_{nt+1}}{f_t R^{si}_{t+1}+(1-f_t)R_{mt+1}}\right] = (1-\pi)\left[\frac{R^{ss}_{t+1}-R_{mt+1}}{f_t R^{ss}_{t+1}+(1-f_t)R_{mt+1}}\right]. \tag{A10}$$

This is equivalent to (24) in the text. Because $q_t > 1 > q^R_t$ in our equilibrium, we always have

$$R^{ii}_{t+1} > R^{si}_{t+1} \text{ and } R^{is}_{t+1} > R^{ss}_{t+1}.$$

In the neighborhood of the steady state equilibrium, we have

$$R^{ii}_{t+1} > R^{ss}_{t+1} > R_{mt+1}.$$

Thus, comparing $(A9)$ and $(A7)$, we learn the inequality $(A7)$ holds in the neighborhood of the steady state.

7.3 Proof of Claim 2

Let l be the ratio of real money balance to capital stock in the steady state. From $(26, 27, 28)$ in the text, the steady state value of (r, q, l) solves

$$r = 1 - \lambda + (1-\beta)\left[r + \lambda(1-\pi+\pi\phi)q + \lambda\pi(1-\phi)q^R + l\right], \tag{A11}$$

$$(1-\lambda)(1-\theta q) = \pi[\beta(r+\lambda\phi q) - \lambda(1-\beta)(1-\phi)q^R + \beta l], \tag{A12}$$

$$(1-\pi)\frac{\frac{r}{q}+\lambda-1}{(r+\lambda q)\chi + l} = \pi\frac{1-\frac{r+\lambda\phi q+\lambda(1-\phi)q^R}{q}}{[r+\lambda\phi q+\lambda(1-\phi)q^R]\chi + l} \tag{A13}$$

where $\chi = \theta(1-\lambda) + (1-\pi+\pi\phi)\lambda$ and $q^R = \frac{1-\theta q}{1-\theta}$ as in the text.

From $(A11)$ and $(A12)$, we have

$$\begin{pmatrix} 1 & -(1-\beta) \\ \pi & \pi\beta \end{pmatrix}\begin{pmatrix} \beta r \\ l \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\kappa + \begin{pmatrix} \lambda(1-\beta)(1-\pi\eta) \\ -\left[\widehat{\theta} + \lambda\pi\beta(1-\eta)\right] \end{pmatrix} q$$

where $\eta = \frac{1-\phi}{1-\theta}$, $\kappa = 1-\lambda+\lambda\pi(1-\beta)\eta$ and $\widehat{\theta} = (1-\lambda+\lambda\pi\eta)\theta$. Thus

$$\pi\beta r = (1-\beta+\pi\beta)\kappa + (1-\beta)\left[\lambda\pi\beta(1-\pi)\eta - \widehat{\theta}\right]q \tag{A14}$$

$$\pi l = (1-\pi)\kappa - \left[\lambda\pi - \lambda\pi(\beta+\pi-\pi\beta)\eta + \widehat{\theta}\right]q. \tag{A15}$$

Because $\lambda\pi - \lambda\pi(\beta+\pi-\pi\beta)\eta + \widehat{\theta} = \lambda\pi(1-\pi)(1-\beta)\eta + \lambda\pi\phi + \theta(1-\lambda) > 0$, we have

$$l > 0, \text{ iff } q < \frac{(1-\pi)\kappa}{\lambda\pi - \lambda\pi(\beta+\pi-\pi\beta)\eta + \widehat{\theta}} \equiv \widehat{q}. \tag{A16}$$

Then there is a monetary equilibrium with financing constraint only if $\widehat{q} > 1$, or

$$(1-\lambda)\theta + \lambda\pi\phi < (1-\lambda)(1-\pi),$$

i.e., Condition 1 in the text for the first best allocation is violated.

$(A13)$ can be written as

$$[r-(1-\lambda)q]\left[r-(1-\lambda)q+q+\frac{l}{\chi}\right]=\lambda\eta(q-1)\left[r-(1-\lambda)q+\pi\left(q+\frac{l}{\chi}\right)\right] \tag{A17}$$

Together with $(A14)$ and $(A15)$, we have the condition for steady state q as

$$\begin{aligned}
0 &= \Psi(q) \\
&\equiv \lambda\pi\eta(q-1)\begin{pmatrix} \left\{(1-\beta+\pi\beta)\kappa+q\left[(1-\beta)\left(\lambda\pi\beta\eta-\widehat{\theta}\right)-\pi\beta\kappa\right]\right\}\left[\lambda(1-\pi\eta)+\widehat{\theta}\right] \\ +\pi\beta(1-\pi)\left[\kappa+q\left(\lambda\pi\beta\eta-\widehat{\theta}\right)\right] \end{pmatrix} \\
&\quad -\left\{(1-\beta+\pi\beta)\kappa+q\left[(1-\beta)\left(\lambda\pi\beta\eta-\widehat{\theta}\right)-\pi\beta\kappa\right]\right\}\cdot \\
&\quad \begin{pmatrix} \left\{(1-\beta+\pi\beta)\kappa+q\left[(1-\beta)\left(\lambda\pi\beta\eta-\widehat{\theta}\right)-\pi\beta\kappa\right]\right\}\left[\lambda(1-\pi\eta)+\widehat{\theta}\right] \\ +\beta(1-\pi)\left[\kappa+q\left(\lambda\pi\beta\eta-\widehat{\theta}\right)\right] \end{pmatrix}.
\end{aligned} \tag{A18}$$

Then we learn

$$\Psi(1)=-(1-\beta)\left(\kappa+\lambda\pi\beta\eta-\widehat{\theta}\right)^2\left\{(1-\beta)\left[\lambda(1-\pi\eta)(1-\theta)+\theta\right]+\beta(1-\pi)\right\}<0.$$

Therefore the necessary and sufficient condition for the existence of monetary equilibrium with $q\in(1,\widehat{q})$ is

$$0<\Psi(\widehat{q}). \tag{A19}$$

Using $(A16)$, $(A19)$ becomes

$$\begin{aligned}
0 &< \beta\lambda\eta(1-\pi-\chi)\left[\chi-\beta(1-\pi)+\pi\beta(1-\pi)\right]-\left[\chi-\beta(1-\pi)\right]\chi\kappa \\
&= \pi\beta^2\lambda\eta(1-\pi)(1-\pi-\chi)-\left[\chi-\beta(1-\pi)\right](1-\lambda+\lambda\pi\eta)\left[\chi-\beta\lambda(1-\pi)(1-\phi)\right].
\end{aligned}$$

Multiplying both sides with $1-\theta$, we get the condition

$$\begin{aligned}
0 &< \pi\beta^2\lambda(1-\pi)(1-\phi)[(1-\pi)(1-\lambda)-(1-\lambda)\theta-\pi\lambda\phi] \\
&\quad +[(\beta-\pi)(1-\lambda)-(1-\lambda)\theta-\pi\lambda\phi]\left[(1-\lambda)(1-\theta)+\lambda\pi(1-\phi)\right] \\
&\quad \cdot\left[\lambda(1-\pi)(1-\beta)+\theta(1-\lambda)+\phi\lambda(\pi+\beta-\pi\beta)\right].
\end{aligned}$$

This is equivalent to Assumption 2 in the text. Therefore, under Assumption 2, we have a competitive equilibrium in which fiat money has a positive value (Claim 3(i) $p_t > 0$) and $q_t > 1$ (Claim 2(ii)). Claim 2(iii) directly follows from inequality $(A4)$ or $(A7)$ which we proved above given that $q_t > 1$ in the neighborhood of the steady state. QED.

7.4 Proof of Claim 3

Claim 3(i):

Under Assumption 2, we have $(1-\lambda)\,\theta + \pi\lambda\phi < (1-\lambda)(1-\pi)$. Thus we have

$$\begin{aligned} \frac{\partial \text{ RHS of } (A11)}{\partial q} &= \lambda\,(1-\beta)\left(1-\pi+\pi\phi-\pi\frac{1-\phi}{1-\theta}\theta\right) \\ &> \lambda\,(1-\beta)\,\frac{\pi\phi}{(1-\lambda)(1-\theta)} > 0. \end{aligned}$$

Given $q > 1$ and $l \geq 0$, we have from $(A11)$

$$r > 1-\lambda+(1-\beta)(r+\lambda), \text{ or}$$

$$r+\lambda > \frac{1}{\beta}.\ QED.$$

Claim 3(iv):

Suppose that 3(iv) is not true. From $(A13)$ and $(A11)$ respectively, we have

$$\frac{r}{q} \leq 1-\lambda$$

and

$$\frac{r}{q}-(1-\lambda) \geq (1-\beta)\left[\frac{r}{q}(1-\pi)+\lambda(1-\pi)+\pi\right]+\frac{1-\beta}{q}l.$$

Together these imply $l < 0$. This is contradiction. QED.

Claim 3(iii):

From equation $(A13)$, Claim 3(iv) implies $\frac{r}{q}+\lambda > 1$. QED.

Claim 3(ii):

Using $q^R < 1 < q$ in $(A13)$, we have

$$(1-\pi)\left[r-(1-\lambda)q\right] > \pi\left[q-r-\lambda\phi q-\lambda\left(1-\phi\right)q^R\right], \text{ or}$$

$$r-(1-\lambda)q > \pi\lambda\eta\left(q-1\right) > 0.$$

Hence, given $l > 0$, from $(A17)$ it follows that

$$\frac{\Delta\left(\Delta+1\right)}{\Delta+\pi} < \lambda\eta\left(1-\frac{1}{q}\right),$$

where $\Delta \equiv \frac{r}{q}+\lambda-1 > 0$ by Claim 3(iii). But from $(A14)$,

$$\frac{1}{q} = \frac{1}{\left(1-\beta+\pi\beta\right)\kappa}\left\{\pi\beta(\Delta+1-\lambda)-(1-\beta)[\lambda\pi\beta\left(1-\pi\right)\eta-\widehat{\theta}]\right\}.$$

Substituting this to the above inequality, we get

$$\begin{aligned} & \frac{\Delta\left(\Delta+1\right)}{\lambda\eta(\Delta+\pi)}+\frac{\pi\beta\Delta}{\left(1-\beta+\pi\beta\right)\kappa} \\ < \ & 1-\frac{1}{\left(1-\beta+\pi\beta\right)\kappa}\left\{\pi\beta(1-\lambda)-(1-\beta)[\lambda\pi\beta\left(1-\pi\right)\eta-\widehat{\theta}]\right\}. \end{aligned}$$

The LHS of this inequality is increasing in Δ.

Suppose Claim 3(ii) is not true, i.e., $\Delta \geq \frac{1-\beta}{\beta}$. Then

$$\begin{aligned} & \frac{1-\beta}{\lambda\eta\beta(1-\beta+\pi\beta)}+\frac{\pi(1-\beta)}{\left(1-\beta+\pi\beta\right)\kappa} \\ < \ & 1-\frac{1}{\left(1-\beta+\pi\beta\right)\kappa}\left\{\pi\beta(1-\lambda)-(1-\beta)[\lambda\pi\beta\left(1-\pi\right)\eta-\widehat{\theta}]\right\}. \end{aligned}$$

Multiplying this inequality through by $\lambda\eta\beta(1-\beta+\pi\beta)\kappa$, we have

$$\begin{aligned} & (1-\beta)\left[1-\lambda+\lambda\pi(1-\beta)\eta\right]+\lambda\eta\beta\pi\left(1-\beta\right) \\ < \ & \lambda\eta\beta(1-\beta+\pi\beta)\left[1-\lambda+\lambda\pi(1-\beta)\eta\right]-\lambda\eta\beta^2\pi(1-\lambda) \\ & +\lambda^2\eta^2\beta^2\pi(1-\pi)(1-\beta)-\lambda\eta\beta(1-\beta)(1-\lambda+\lambda\pi\eta)\theta. \end{aligned}$$

Cancel the two terms which do not have factor $1-\beta$, and divide by $1-\beta$, we get

$$(1-\lambda+\lambda\pi\eta)\left[1-\lambda\eta\beta\left(1-\theta\right)\right] < 0, \text{ or}$$

$$(1-\lambda+\lambda\pi\eta)\left[1-\lambda\beta\left(1-\phi\right)\right] < 0.$$

This is contradiction. QED.